The Mathematics Teacher

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The impact of modern mathematics on secondary schools[1]

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A paper on a vacuous subject, but nonetheless one of great importance for the high-school program.

MY SUBJECT is vacuous; the lively modern development of mathematics has had no impact on the content or on the presentation of secondary-school mathematics. Algebra and geometry, as covered in schools, consist exclusively of ideas already well known two hundred years ago—many of them two thousand years ago. No matter how much better the teaching of these particular ideas to more and more pupils, their presentation leaves school mathematics in a state far more antiquarian than that of any other part of the curriculum. The pupils can conclude only that there is no such thing as a new mathematical idea.

No conclusion could be further from the truth. Modern mathematics is full of new ideas, of new uses of old concepts, and of solutions to old problems, as witness for example the spectacular solution by three Americans in 1953 of the famous "Fifth Problem" of Hilbert (1900). Except for physics, none of the standard academic subjects has today more intellectual life and drive than mathematics. That the secondary schools have isolated themselves from this life is no one's fault and everyone's negligence. The mathematicians have pursued the new ideas with single-minded devotion; the teachers have concentrated on the reformation and presentation of the old ideas; there has been no exchange of interests.

Without an exchange of ideas, the impact of modern mathematics on the schools and society will be delayed and the result disastrous. The growing applications of mathematics in technology and in science call for many more well-trained young engineers; this training requires good mathematics in the schools. The need for a new generation of teachers of mathematics and science in the secondary schools is even more desperate. The growth of research in industries and in government laboratories has called for more research mathematicians. Before 1940 practically all professional mathematicians were engaged in teaching and in academic research; now more than twenty per cent of these mathematicians have positions in industry or government. One field alone, the management of the new high-speed digital computing machines, will call for hundreds of new mathematicians.

In purely academic positions there is also a grave shortage of new mathematical talent. Today the United States is one of the two or three leading countries in the development of pure mathematical research, but this position has been achieved only thanks to the presence here of many talented mathematicians from Europe. At each of four leading American institutions for pure mathematics, over forty per cent of the full professors are men with European backgrounds. For the future, it is essential that we develop new American talent. In the secondary schools, this means that young Americans with possible talent

[1] Reprinted from *The Bulletin of the National Association of Secondary-School Principals*, XXXVIII (May 1954).

should have an opportunity to learn what mathematics is today, and not what it was for the Greeks.

But what modern mathematical ideas are relevant for pupils in the secondary schools? Take algebra, which deals primarily with the properties of addition and multiplication. A simple and pervasive idea of modern algebra is this: addition is not just the usual addition of numbers. Everyone knows how to add numbers; everyone also knows how to add hours. Six hours past eleven o'clock is five o'clock; three hours after ten o'clock is one o'clock. These sums may be written as: $6+11=5$ and $3+10=1$. This is not an ordinary sum. It is rather the ordinary sum after 12, and its multiples are discarded: $6+11=17$ and $17-12=5$. The resulting process is known as addition "modulo 12"; in case one joins the Navy, one learns its counterpart, addition "modulo 24." The same sort of addition can be carried out with any positive whole number (in place of 12 or 24) as the modulus, and the process of addition which results has all the formal algebraic properties of ordinary addition.

For example, for any "numbers" a, b, and c added modulo 12, we have the rules: $a+(b+c)=(a+b)+c$; $a+b=b+a$; and $a+0=a$.

The conclusion is that algebra is really the study of the formal properties of addition and multiplication, like those listed just above for addition. These formal properties lead to the notion of an Abelian group as a collection of things (not necessarily numbers), which can be added subject to these and other formal properties. This notion of a group is a pervasive one; it appears throughout mathematics, in geometry and in mechanics. The notion of a group provides an analysis of the esthetic idea of symmetry. Finally, the basic description and illustration of this notion could easily be made accessible to high-school students as one instance of a simple and exciting new mathematical concept.

Take proofs or demonstrations. These have always appeared in school geometry, sometimes in fear-provoking form, and they have traditionally disappeared in school algebra, which is viewed more as a sort of manipulation combined with the deciphering of mysterious unknowns. The fact of the matter, as emphasized by the modern development of precise and rigorous mathematics, is that proof is the form in which all mathematics appears, be it geometry or algebra or calculus. Once algebra is properly recognized as the study of the formal properties of addition and multiplication, it becomes clear that algebra is concerned with the proof of new formal properties from given ones—and that these proofs are simpler and much more perspicuous than those of geometry.

Customarily, some of the proofs of algebra are buried in geometry, where they suffer from the limitations of the classical Greeks, who favored geometry and did not have algebra. How many pupils still labor through cumbersome statements like, "if equals are added to the same thing, the results are equal," when they should be dealing with the simpler modern statement: "If $a=b$, then $a+c=b+c$." The trouble lies in part with recent textbooks for schools, which sometimes are so obsessed with the hope of using proofs and logical arguments in practical matters that they lose sight of the simple ways of using proofs where they really belong: throughout mathematics.

Take arithmetic, or rather, let's take arithmetic without counting to ten on our fingers. This method of counting is responsible for our use of the ten "digits" 0, 1, 2, . . . 9, and for the fact that the symbol 13 means 10 plus 3, while 245 means: $245=2(10)^2+4(10)+5$. Instead, let's use only two digits and count 1, 10, 11, 100, 101, instead of 1, 2, 3, 4, 5. Then, for example, 1011 will mean: $1011=2^3+2+1$, or eleven, in the ordinary notation. This idea of a different base for numbers is by no means new; but it has

new importance, because it is the method of counting used of necessity in the powerful modern digital computing machines. The reason is that these machines count on electrical relays, which have only two "fingers," off and on. This idea has the advantage of simplicity and practical relevance—plus the striking feature that it promotes real understanding of what the ordinary symbols for numbers mean. It is a notion that should be introduced to every secondary-school pupil, whether in business arithmetic, algebra, or elsewhere.

Take geometry. The Greeks understood the geometry of our ordinary three-dimensional world and reduced the geometry of this space to axioms and proofs. Today, pupils sometimes still labor elaborately through the proofs of this solid geometry in spite of the fact that most of these ideas can now be better developed by means of analytic geometry (the subject that ought to be in the curriculum at this point). The important modern discovery is that space is not just the three-dimensional Euclidean space of our immediate experience, but that spaces and geometries of the utmost variety and character are possible and useful. For instance, most surfaces have two sides, but there can be one-sided surfaces. An easy illustration can be achieved by taking a long strip of paper, turning one end over, and then pasting the ends together. This model is but one vivid illustration of the notions of a new part of geometry known as topology. Again, take the notion that space has three dimensions, as illustrated by the fact that it takes three numbers to tell where you are. Thus you may be on the 7th floor at 579 East 12th Street, and the three numbers are the distances east, north, and up from the center of the city. But to tell where you are and where you are heading takes six numbers—two more to specify your direction and still another for your speed. This means that "space" of positions and velocities has six dimensions. It is but one illustration of the way in which the modern idea of the variety of possible geometries can be presented.

Take trigonometry. When taught, it still appears chiefly as a means of solving triangles, and these triangles are often solved laboriously with logarithms, even though in actual practical use the logarithms might often be replaced by computing machines. The real idea behind logarithms is often lost, although it can be stated very simply in symbols as a method of changing multiplication into addition: $\log(ab) = \log a + \log b$. This single equation, and not the long computation, is what really matters in higher mathematics, for the self-same idea of turning multiplication into addition appears over and over again in group theory and elsewhere, where it goes under the general name of a homomorphism (of a multiplicative group changed into an additive group).

The trigonometric functions are likewise put in for the wrong reaons. The function $\sin x$ is not important because of long tables of its values or because of fancy identities; it is important because it is a function which repeats itself periodically according to the equation $\sin(360° + x) = \sin x$. It is this behavior which makes the trigonometric functions important in the analysis of all sorts of waves and other periodic or recurring phenomena from business cycles and the equations of heat flow to the most sophisticated parts of modern harmonic analysis.

Take calculus—or should we take it? In this country it is traditionally a subject for colleges, where it is often shrouded in the undeserved reputation of undue difficulty, while in Europe it traditionally belongs to the secondary schools. That, I submit, is where it really belongs, at least in its initial stages. The idea of a derivative is a basic one in calculus. The derivative can appear as a velocity; indeed, only in this language can one properly formulate this basic notion of physics. The derivative can be interpreted as a marginal utility, and here it gives the proper formulation to this basic notion of economics and

social studies. Modern technical society would be impossible without the devices provided by the calculus; it is high time the secondary-school pupil has some opportunity to see what this means.

Take the algebra of sets as an illustration of one of the still newer mathematical subjects of relevance to the schools. A set A is just an arbitrary collection, finite or infinite, of things. There are algebraic operations on sets A and B. The sum $A+B$ is defined as the set of those things which belong either to A or to B or to both; the product AB is the set consisting of those things which belong to both sets A and B. These operations are not quite like those of the ordinary algebra of numbers because for sets we have both $A+A=A$ and $AA=A$, but there is nevertheless a consequential algebra of sets. The very simplicity and generality of the notions involved account for their wide applications—from logic and the design of electrical circuits to various social studies. Furthermore, this algebra of sets could readily be made accessible to pupils in the secondary schools.

To summarize, modern mathematics is marked by the construction of a wide variety of new concepts, by the abstract consideration of their properties and the concrete examination of their instances, and by the formal demonstration of facts about them. The examples above give but a few instances of some of the concepts that can make sense to pupils in the secondary schools. The mathematical subjects now taught there need drastic overhauling. It no longer suffices to continue to teach the old ideas; rather, one must start afresh to determine which ideas should be taught and in what perspective. In this way, school mathematics could become fully relevant to the modern world.

Have you read?

BOYER, CHARLES E. "Tighter Disciplinary Policy Necessary for Better Education," *The American Teacher Magazine*, February 1955, pp. 11–12.

All of us have discussed the problem of discipline from the class and school point of view. The American Federation of Teachers therefore took it upon themselves to make a survey of the problem of discipline. Questionnaires were sent to all state departments of education and to all presidents of A. F. of T. locals. While recognizing the limitations of the questionnaire method the results are still worthy of consideration. Some items of interest are: 90% of A. F. of T. presidents thought a tightening of discipline was necessary and would improve instruction; only one state prohibits corporal punishment; only four states have set up some limitations. Several states authorize the teacher's use of corporal punishment. There are, however, many school corporations where corporal punishment is prohibited.

There are many other comparisons which will interest you, whether or not you agree with the implications or with the conclusions. You will be interested in the reason for accepting a philosophy which gives authority to discipline for the sake of the child and for the good of the group. You should read this article and draw your own conclusions.

"Report of the Joint Committee of the American Society for Engineering Education and the Mathematical Association of America on Engineering Mathematics," *The Journal of Engineering Education*, April 1955, Vol. 45, No. 8, pp. 583–588.

Although this report was made for engineering schools to use, it has much information of value for the secondary mathematics teacher. The article answers three questions: 1. What should be taught? 2. Who should teach it? 3. How should it be taught?

You will be interested to note that the answer to the first question includes such things as computing machines, binary representation, precision measure and computation, sampling, statistics, slide rule, and graphical representation. The colleges recognize that high-school students entering engineering schools will not all have strong mathematics backgrounds, but they hope that in the future these backgrounds will be strengthened. The *how* will interest you in that it maintains more emphasis should be given to graphical representation, numerical methods, fundamental concepts, and more use of mathematics in related courses. By careful study of this article we can help pave the way for better engineers.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

The problem of varying abilities among students in mathematics[1]

WILLIAM D. REEVE, *Emeritus, Teachers College, Columbia University, New York, New York.*

How to provide for varying abilities in a mathematics classroom is a perplexing problem. How far can the secondary schools go in providing a variety of courses for pupils of different abilities and interests?

THE UNITED STATES now has more children above 14 years of age in school than all the other countries in the world combined. The most recent figure of the enrollment in public secondary schools is 6,168,000 in 1951. Secondary high school enrollment has grown about 5 times as fast as the population in general. The fact that between 60 and 90 per cent of the population of eligible age in a community is enrolled in the secondary school means that the students are not so select as they were 50 years ago when only about 500,000 children received anything beyond an elementary education. Just imagine then how some teaching problems would be complicated if everybody were in school! As it is, teachers insist that the problem of the slow-learning student is almost impossible to solve. Many of them do not realize that the problem of the superior student is more difficult. The American public school system is unique in world history. Its students are certainly the most schooled of any group in the world, but are they as well educated?

The most important and most difficult problem today is the problem of individual differences among students. They differ not only in ability, but in their experiences and interests. It has taken the educational world a long time to appreciate fully the fact that great differences among individuals exist, and, in many respects, we are not yet fully aware of their significance. Three questions face us at the outset when we attempt to consider this problem. First, "What are the best methods of discovering these individual differences?" Second, "How great are these differences?" Third, "What are the best methods of handling these differences, once they are discovered?"

A great many teachers, sooner or later, discover very marked differences in ability among students of varying chronological ages in the same class, and also between students of the same chronological ages in the same class, but often these differences are not discovered for some time after the teacher takes charge of the class. This is especially true nowadays, where, in so many schools, the classes are seriously overcrowded. We must make a serious and determined effort to solve the problem, or run the risk of enormous losses in the educational output of the next generation. The problem of making every

[1] Paper read at the Winter Meeting of the Association of Mathematics Teachers of New Jersey at Rutgers University in Newark, N. J., on Saturday, March 5, 1955; at Illinois State Normal University in Normal, Ill., on Saturday, March 26, 1955; at the Houston Mathematics Council Meeting at Houston, Texas, on May 4, 1955, at the Rutgers University Mathematics Institute on July 16, 1955; and also at the University of Wisconsin Summer Conference of Mathematics Teachers in cooperation with the National Science Foundation, at Madison, Wis., on July 21, 1955.

child the right kind of citizen is largely a problem of education. Each child should be given the opportunity to exercise his ability in the way that will permit him to achieve his fullest development. This cannot be done in the majority of classes in the schools of this country as they are organized today. It is this failure to recognize and to handle individual differences of ability among students, that gives rise to the large and unwarranted number of failures in the schools.

It is absurd to try to teach the best 10 per cent of a normal distribution of students in the same class with the poorest 10 per cent, without doing great injustice to both. I do not believe that it is possible to teach such groups together *at all* in the best sense. The present practice of teaching such varying types together can lead only to educational bankruptcy. It is inevitable that certain adjustments must be made. We would already have made more progress in solving the problem if it were not for the unfortunate attitude of certain educators toward homogeneous classification for teaching purposes. For example, at a recent meeting of mathematics teachers, I heard a supervisor from one of the largest American cities say that the discipline problem in his schools was being solved by putting the gifted students in the same class with the slower ones. This is a good way to stimulate discipline problems if I know anything about it. The gifted student loses his chance to develop himself to the fullest extent and the slow-learning student becomes discouraged and gives up the ghost.

If we are to avoid all of the embarrassment of such situations as those described above, we shall have to formulate, or at least implement, a better *guidance program* in the schools. In the interest of the nation and in protecting society's stake in the educational enterprise, at least a goodly share of the more brilliant youth should be directed into government service, into scientific work, or business leadership, and appropriate training should be provided.

In speaking about "Provision for individual differences" in his department, "Memorabilia Mathematica," on pages 39 and 40 of the January, 1955, issue of THE MATHEMATICS TEACHER, William L. Schaaf, of Brooklyn College, quoted the first two paragraphs of a brief report on "Mathematics in Secondary Schools," issued by the Scottish Education Department and published in Edinburgh in 1950, which said:

"Mathematical studies have long held in the school curriculum the assured place which is due to one of the most distinctive activities of the human mind, of fundamental importance in both the intellectual and the practical field. Pure mathematics is, however, highly abstract and formal in character, and it is natural that these characteristics have given rise to difficulties in the school treatment of the subject. The traditional treatment has been not ill-suited to the ablest pupils, who have shown their capacity to profit from the discipline it can afford, but for the less well-endowed the academic rigours must be tempered. Recognition of this necessity has already found practical expression in many schools which have adapted courses to the varying capacities of their pupils. This process of adaptation, however, has not yet been applied widely enough, and the need for a greater differentiation of courses has become still more pronounced with the general acceptance of the principle that promotion to the secondary stage should in the main be determined by age rather than by attainment.

"Differentiation has taken the form not only of adjusting the various syllabuses to suit the capacity of the pupils, but also of modifying their content in varying degrees to relate them, as appropriate, to the needs of life. Such practical references are not only important in themselves but they can also be a powerful factor in enlisting the interest of those who do not respond readily to a more academic approach. Thus, in the syllabus intended for the weakest pupils, mathematics

covers little more than the simple arithmetic of everyday life; in the syllabuses for the average pupils a treatment is encouraged which would relate mathematics to the life of the adult world in its technical, commercial, domestic, and civic aspects; and in the syllabus for the ablest pupils it is sought to combine this broader approach with the more rigorous treatment required as a foundation for advanced study."

Schaaf then adds a final comment, thus:

"So it would seem that learners differ from one another in much the same way everywhere, and that the teacher of mathematics is beset with the problem of providing for individual differences equally on either side of the Atlantic. One calls to mind the universal nature of mathematics; it appears that mathematical education, also, has about it an aura of universality."

It is interesting to note what Field Marshall Montgomery said recently in the Sachs Lectures where he spoke on "Education for Leadership" before the sixth annual alumni conference of Teachers College at Columbia University on November 22 and 23, 1954. According to *Newsweek*, December 6, 1954, he said:

"What is the object of education? In my view it is to develop the mind, the will, and the conscience of a boy and to strengthen his character. To educate is to train mentally and morally. A good educational system should result in what has been called the noblest work of God—an honest, trustworthy, and chivalrous man. . . . If this can be achieved, we are well on the way to laying the foundations of leadership. . . .

"The whole idea of 'leadership' is regarded with deep suspicion by certain influential sections of Western social opinion. . . . It follows that little can be done until Western opinion is re-educated to appreciate the crucial necessity of devoted and able leadership in all aspects of democratic life. We must train heads as well as count them. . . .

"How can we ensure that among the great mass of boys in any nation, those who have been endowed with superior talents will have the opportunity to develop [those] talents to the full. . . . How can we ensure that the less gifted will receive that education which is their just due? . . .

"Some form of selection, some sort of 'weeding' is necessary if a nation is to get good results from the education and training of its boys. . . . If leaders are not selected on sound principles, they will merely select themselves on unsound principles—by their birth, or wealth, or influence—and if they are not educated for their responsibilities, their leadership will not be more democratic, but merely less wise. If the right of the well-endowed . . . is denied—the well-endowed with ability, not . . . with dollars . . . then the rich or the well-born or the demagogues or the toughs will become the leaders. These are the only alternatives."[2]

In a letter to the *New York Times* dated March 29, 1954, J. H. Landman said: "Millions of dollars are expended by the City of New York annually for the education and care of idiots, imbeciles, morons and other subnormals in our schools and so little money for our superior students. What a travesty on our school system!"

Landman goes on to point out the number of normal, subnormal, and superior children in grades 1–12 and how little is being done to give proper attention to exceptional students who should be the future leaders of our country.

[2] See *Teachers College Record*, January, 1955, pp. 181–202 for the complete lectures. See also Alice M. Reeve, "Planning a Mathematics Program for the Gifted Pupil in Senior High School," *New York State Mathematics Teachers Journal*, April, 1955, pp. 9–12; A. Harry Passow, Miriam Goldberg, Abraham J. Tannenbaum, and Will French, *Planning for Talented Youth*, Bureau of Publications, Teachers College, Columbia University, 1955; and *College and University Programs for the Preparation of Teachers of Exceptional Children*, Bulletin No. 13, 1954, U. S. Department of Health, Education and Welfare, Office of Education; Earl M. McWilliams and Kenneth E. Brown, *The Superior Pupil in Junior High School Mathematics*, U. S. Department of Health, Education, and Welfare, Office of Education, Bulletin No. 4, 1955.

Terman recently pointed out[3] that, when he began his original studies, "child prodigies were in bad repute because of the prevailing belief that they were usually psychotic or otherwise abnormal and almost sure to burn themselves out quickly or develop post-adolescent stupidity." Terman's work has disproved this supposition and has exposed its injustice.

Terman further points out: "The exceptionally bright student, who is kept with his age group, finds little to challenge his intelligence and all too often develops habits of laziness that later wreck his college career."

Improving educational methods for gifted students is undoubtedly one of our main educational problems. According to *Time*, December 13, 1954, classicist John Francis Latimer, of George Washington University, said that he has evidence to prove that the high school in the United States has "embarked on a retreat from solid learning." He took several polls, one of which showed that "In 1900 at least four out of five students took mathematics. Today fewer than half the students do, and of these, 13% are taking 'general mathematics' which Latimer calls a preparation for nothing, as far as college is concerned. Biggest slump: algebra, down from 56 per cent to 20 per cent." Latimer further points out what should be obvious to the alert educator: "Group action subjects which the student can often bull his way through without cracking a book" are taking the place of traditional subjects.

He goes on to say: "We have gotten away from individual effort. . . . By permitting the high schools to become the vocational bargain basements of education, we have insulted the students' intelligence and encouraged mediocrity by prescribing mediocre subject matter for mediocre minds."

Can anyone really deny this?

On page 14 of the *Saturday Review* for December 11, 1954, William Barrett, in reviewing *The Age of Conformity* by Alan Valentine, said:

"The idea of democracy is sometimes thought erroneously to be the glorification of the Common Man. But Thomas Jefferson, the American father of the democratic idea, extolled the *capacities* of the common man, not his achievements, underlining thereby the necessity of education, discipline, and leadership if these capacities are to be realized.

. . . .

"Where is the remedy? How can we halt this mass drift toward conformity? Since we are committed to democracy, Mr. Valentine holds, the solution we are to look for has to be a democratic one: the people themselves must decide what their democracy is to mean and what their freedom is to be used for. Once and for all, it must be realized that equal opportunity does not mean equal capacity, and that elite groups, democratically recruited from the whole body of society, must be given their proper place and prestige in American life. Otherwise our democracy is bound to sink further and deeper into the swamp of human mediocrity."

In an article, "Education for All Is Education for None," in the magazine supplement of the *New York Times* for January 9, 1955, Professor Douglass Bush of Harvard said:

"In schools, colleges and universities today, the results of the huge increase in the student body suggest a rather painful thought: the principle of education for all, however fine in theory, in practice ultimately leads to education for none. In other words, the ideal of education for all forces acceptance of the principle that the function of education is primarily social and political rather than purely intellectual; if school standards are geared to an almost invisibly low average, there is not much real education available for anyone, even for the gifted.

. . . .

[3] Lewis M. Terman, "Child Prodigies Do Make Good," *New York Times*, March 28, 1954.

"As things are, we have an army of misfits, who lower educational standards and increase expense, and no branch of a university staff has grown more rapidly of late years than the psychiatric squad.

"Secondly, many people have grounds for the belief that the multiplying junior colleges can and will drain off a large number of the young who, for various reasons, are unfitted for a really strenuous four-year course. Junior colleges, however, should not be recreational centers for the subnormal.

"Thirdly, I think the need for formal education beyond high school would be much lessened and the quality of both secondary and higher education obviously raised if the colleges and universities, getting the public behind them, made a concerted effectual demand that the schools do their proper work and do it better than a great many schools have been doing it.

"Quite commonly, a distressing proportion of a college course now consists of high school work. For example, we have grown so accustomed to a battalion of instructors teaching elementary composition to freshmen that we take it as a normal part of college education, whereas, in fact, it is a monstrosity. Imagine a European university teaching the rudiments of expression! If high school graduates are illiterate, they have no business in college. For a long time and for a variety of reasons we have had slackness all along the line; somehow, some time, strictness and discipline have got to begin."[4]

In an article on the first page of the *New York Times* for May 23, 1955, the crisis in mass education is aired by the Carnegie Corporation of New York in its final report. Among other things it said:

"John W. Gardner, the president, warned that a crisis confronted higher education in this country as a result of the growth of college and university enrollments. They are expected, in the Nineteen Sixties, to be nearly double the figures of today.

"The 'intolerable pressure' this would put on institutions of higher learning, Mr. Gardner said, requires that those interested in education begin now to seek answers to the following questions:

1. Who should go to college?
2. What kinds of education should be provided?
3. How can we avoid the worst effects of 'mass production' in education?
4. How shall we pay for it all?

"In an analysis of the questions, Mr. Gardner said the corporation was not attempting to provide answers but 'to state the issues and to suggest terms of discussion.' "

This report is a very important one and it should be read by all educators.

The trouble with us is that we have adapted the program in mathematics to mass education with the natural result that we have produced a lowering of scholarship. Most, if not all, of the examinations in recent years have reduced standards until what we used to expect of students in the ninth grade, we are glad to have them attain in the eleventh grade. To be sure, examinations should be made simple and easier for certain types of students, but not for all!

Obviously, we cannot ignore the needs and desires of the 6,180,000 and more students who attend the secondary school today. But do we know what they are? Granted that we do, are we then ready to adjust the subject matter to their individual differences? In France, for example, the aim is to train mathematicians, but that cannot be a broad aim in this country, even though we shall probably train a few. This is not only desirable for the many but necessary, because the student must also learn something of the other great fields of knowledge such as language, science, practical arts, the social studies, and the fine arts.

[4] See also the next to the last page of the *New York Times Magazine Supplement* for a large number of letters to the *Times* editor, protesting Bush's stand. And see also Bush's final letter to the editor of the *Times* in the *Times Magazine Supplement* for January 30, 1955.

Many plans for helping us to solve some of the pressing curriculum problems have been suggested by educational leaders.[5] Among them are:

1. *Work outside the classroom.* One way to keep the brighter students busy and avoid loss for them is to assign outside work for them to do at home or in the library. In addition to being kept busy, the more gifted students can learn other things of great value. For example:

a) They can learn to read better. Too many students are lacking in the ability to look up ways of doing things and methods of procedure in texts other than those they are using. They can learn to give a decent report if some help is given. In fact, some students have had a complete course in algebra in the high school and, at the end of that time, do not know the name of the author of the textbook they are using. For that matter, some teachers don't know it either! Some students do not appreciate the value of reading the preface, and I know a very prominent and intelligent man who said that he studied Latin grammar all the way through a certain text and yet never understood that there was an index in the book. Such a state of affairs would seem funny if it were not really tragic!

b) A student who is given outside work to do, that is of high grade or even above the average of the class, is inclined to develop higher ideals of work and achievement. We need more leaders, and this is one way to start their training. Moreover, we need trained leaders now more than we have ever needed them in our entire history.

c) A bright student may be assigned the task of looking up or working up various methods of proof for the same theorem. I know one high school boy who worked out forty-five distinct proofs for the Pythagorean theorem. Some of the proofs were in French which he read easily.

d) Another good way to keep the bright student busy is to give him some form of enrichment material.

2. *The study class.*[6] Then, of course, there is the *conference hour* or *study class* to which all those students who are behind in their work should be expected to come. This class need not be designated as a class for the slow ones only, and it is probably a mistake to do so, although the chances are that the slow students will receive the most good from it by far. It should be a class where any student may receive help if he needs it. It may be that a very strong student will be absent on account of illness and will need to come to the study class once or twice after his return in order to get some of his ideas set right. Certainly, much good can come from such a class if it is conducted along this line. The study class should not be considered a penal institution although some students may be forced to attend by the teacher. Most of them should come of their own free will, because they realize they need it. Every student should look upon this class as an opportunity not to be overlooked. The failures in any mathematics department can be materially decreased by the employment of this method of caring for individual differences.

I have found the more gifted children taking advantage of the opportunities offered by the directed study hour, and there is no doubt that they profited by it.

[5] Joseph Jablonower, "Recent and Present Tendencies in the Teaching of Algebra in the High Schools," *The Teaching of Algebra*, The Seventh Yearbook, The National Council of Teachers of Mathematics, 1932, pp. 5–12. See also *Teaching Rapid and Slow Learners in High Schools*, Bulletin No. 5, 1954, U. S. Department of Health, Education, and Welfare, Office of Education, U. S. Government Printing Office, Washington 25, D. C., price 35¢; Ruth Strang, "The Mental Diet of Our Gifted Children," *N.E.A. Journal*, May, 1955, pp. 265–266; and *General Education in a Free Society*, Report of the Harvard Committee, Harvard University Press, Cambridge, Mass., 1945, pp. 164–167.

[6] For a complete discussion see William David Reeve, *Mathematics for the Secondary School* (New York: Henry Holt & Co., 1954), pp. 140–148.

3. *Homogeneous classification of students.*[7] Homogeneous classification of students according to ability is the first step toward individualized instruction. Homogeneous classification can be made on the basis of previous school marks or by one of the several good tests available. Some of these may be more reliable than others, but it is easy to get a simple test that will separate the gifted from the slower students with fairly accurate results.

Some argument has always been presented against such classification. One of the main arguments advanced has always been that such a division robs the slower students of much of the spirit of emulation that comes from seeing the achievements of the brighter students, and that parents, as a rule, would rather have their children in classes with the brighter students because of the good results that are to come from such association. The charge is also made that homogeneous classification is undemocratic.

This sounds all right, but is it really true? I have made such a division in classes in mathematics, and, although there was some little objection at first for the reasons set forth above, it was soon evident that all the students gained by the plan. One girl in particular, who was placed in the slow group and who had been extremely reticent and inefficient in the original group, began to make great progress and in a short time was a leader in the slower group. When asked how it was that she now took so much interest in mathematics where before she seemed uninterested, she replied, "You see, when I was in the original group, Ernest and those other bright ones were so good and I felt so inefficient that it made me nervous and afraid to recite and, as a result, I became uninterested. Now that the bright ones are gone I feel that I really know as much as the others, and I am not afraid." The idea that the slower students are helped by being in a class with the brighter students will need modification if the facts are carefully considered. The facts do not agree with personal opinions.

In its "First Report" the Commission on Post-War Plans said:

"We sit in endless meetings discussing the question: shall we give special attention to the bright or to the dull? It is not an either-or proposition! The sensible thing is to provide good courses with very different goals and different experiences for groups with different needs."[8]

Just as the gifted students often have to mark time, so the slow-learning students are often expected to go at the pace set by the brightest ones. Both practices are bad.

4. *Advantages of homogeneous classification.* Now what are the advantages of homogeneous classification of students? Without assuming to have discovered all of the advantages, my experience leads me to suggest the following:

a) The teacher knows fairly well at the outset which of the groups is the stronger, and he can therefore adapt his methods of instruction to each section as a whole. This enables each section to make as much progress as possible and helps the slower ones to go at a pace that will insure learning that is worthwhile. This makes for efficiency in instruction and time.

[7] See *High-School Methods with Superior Students* and also *High-School Methods with Slow Learners*, National Education Association Research Bulletins, Vol. XIX, No. 4, 1941, and Vol. XXI, No. 3, 1943, respectively; "The First Report of the Commission on Post-War Plans," The Mathematics Teacher, Vol. XXXVII (May 1944), pp. 227–232; Howard F. Fehr, "General Ways to Identify Students with Scientific and Mathematical Potential," The Mathematics Teacher, Vol. XLVI (April 1953), pp. 230–234; *The Place of Mathematics in Modern Education*, Fifteenth Yearbook, The National Council of Teachers of Mathematics, 1201 Sixteenth St., Washington, D. C.; See also *General Education in a Free Society*, *op. cit.*, pp. 164–167, for an excellent discussion of what to do with the abler students; W. D. Reeve, *op. cit.*, pp. 59–62; Ruth Strang, *op. cit.*, pp. 265–266, and *Creed for Exceptional Children*, U. S. Government Printing Office, Division of Public Documents, Washington 25, D. C., price 5¢.

[8] "The First Report of the Commission on Post-War Plans," The Mathematics Teacher, Vol. XXXVII (May 1944), p. 228.

b) As a corollary to a) above, homogeneous classification reduces the number of failures, because the students who might otherwise fail are given individual attention earlier in the course and their chances of passing the course are materially increased.

c) On the other hand, homogeneous classification makes it possible for the gifted student[9] to cover a great deal more work in one year than would be possible otherwise. This gain is very important when we consider how great the need for well-trained men and women is today.

d) The most important advantage of homogeneous classification is the enormous gain for society through the certain conservation of human resources.

5. *The problem of the smaller school.*[10] In certain schools we shall be able to classify students in homogeneous groups; but in many schools, especially the smallest ones, this will be impossible. This situation presents a problem which challenges the best thinking of the most highly trained and experienced teachers of mathematics. In the end it means different amounts of possible material for the varying degrees of ability, if indeed it does not mean different courses altogether insofar as quantity and quality are concened. As a result, we may have a course of so-called minimum *essentials*, beyond which we shall add material that will enrich the course for those who are able to do more intensive and extensive work. The alternative will be to permit students to progress as rapidly as possible.

6. *Large-size classes.*[11] The movement for large-size classes never made much headway with a great many teachers, but it is still too much with us. The problem needs further study.

7. *Impetus to work.* We need to remember that the traditional method of assigning the same amount of work to each student may obviate the doing of several times that amount of work by the gifted students. This larger amount of work the gifted students would do only if the proper stimulus were present.

In forming right habits of study it is tremendously important to instill into the students the love of work. This does not mean project method, or problem method, or any method, so much as the right teacher. When live tasks of some importance are set before students in an attractive way, they respond with zeal and interest.

One reason time is lost in teaching mathematics is that we do not know how long it takes to gain a desirable mastery of any topic. This question might be answered by actual experiment. If this information were available, teachers would not continue to do so much teaching beyond the stage of diminishing returns.

According to the *New York Times* of Sunday, May 22, 1955:

"Better understanding of how to educate talented children in public schools is the aim of a study started by the Horace Mann-Lincoln Institute of School Experimentation at Teachers College, Columbia University. The study is designed to help public schools do a better job of identifying and educating youngsters who show special talent.

"The study is being made to help stop the waste of potential talent for business, industry and the arts and sciences. Although much research has been done in

[9] M. L. Hartung, "High School Algebra for Bright Students," The Mathematics Teacher, Vol. XLVI (May 1953), pp. 316–321; D. C. Johnson, "Let's Do Something for the Gifted in Mathematics," The Mathematics Teacher, Vol. XLVI (May 1953), pp. 322–325; *Education for the Talented in Mathematics and Science*, Bulletin No. 15, 1952, U. S. Department of Health, Education, and Welfare; and Ruth Strang, *op. cit.*, p. 265.

[10] H. Vernon Price, "The Small High School," The Mathematics Teacher, Vol. XLV (October 1952), pp. 406–407. See also Marian Schiefele, *The Gifted Child in the Regular Classroom*, Bureau of Publications, Teachers College, New York, N. Y., 1953.

[11] Ellsworth Tompkins, *Class Size—The Larger High School*, Circular No. 305, Federal Security Agency, Washington, D. C., 1949.

this area, relatively few studies have dealt specifically with how public schools can educate children believed to be talented. The Teachers College researchers believe that good programs for such young people can be developed in the public school system."

It seems to me that it would be better to adopt the term "alternate courses" for those courses that may be offered to students with varying ability and interests, so as to avoid the stigma that is at present attached to courses labelled "first track" and "second track." After all, a student may be very intelligent, and yet be not at all interested in a scientific course, planned for those students who intend to be engineers or who plan to work in fields where a great deal of mathematics is involved.

The demand for alternative mathematics courses in the secondary school program is due to the wide differences that exist in the work of the seventh and eighth grades, whose work in mathematics has been limited too much to arithmetic, and to the varying purposes of mathematical instruction in grades 9 to 12.

The course for one group of students may have included only arithmetic, or general mathematics like that set forth in the first or second book of a modern junior high school series. Another group may have taken courses that would prepare them to go to college or technical schools. And other students who intend to go into business or enter upon some other career may have had a different type of training. It would be impossible for the traditional courses of study to meet the needs of all such groups.

Due to the fact that there is now no uniform practice beyond the eighth grade insofar as mathematics is concerned, it is difficult to say just what courses will or should be offered in the ninth and succeeding years. It is clear, however, that certain trends are discernible.

1. *Courses for future mathematicians.* For those students who plan to be mathematicians, there will probably be offered certain courses that will be largely informational and basic. Chief among these are algebra, demonstrative geometry, trigonometry, solid geometry, college algebra, analytic geometry, and the calculus. The same results can be accomplished by a good course in general mathematics as I define it,[12] but not as it is often carelessly defined.

2. *Courses for vocations.* Courses will also be offered in some senior high schools, for those who are preparing to enter certain vocations. Thus, we may find certain schools offering a course in mechanics for those so inclined, a course in commercial algebra or in commercial arithmetic, a course in social-economic arithmetic, a course in statistics, and possibly a course in machine-shop mathematics.

3. *Mathematics for non-academic students.* There is already a strong movement to set up a course for non-academic students in both the secondary schools and in the colleges. The spirit of all of these courses is pretty much the same, but the places where they appear in the course of study and the contents of each course may be different.

Finally, there are some college courses in mathematics arranged for those students (usually freshmen) who do not plan to enter mathematical or scientific careers. Their content material differs more widely from the other non-academic courses I have just described, and in all such courses college credit is given. Here again, it would seem to be better if such work could be offered in the senior high school where better teachers, on the whole, are available and where there would be no question as to giving credit for passing work in the course.

[12] W. D. Reeve, *op. cit.*, chapter 10.

Problem-solving[1]

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The climax to problem-solving is that stage in the process when "all the pieces fall into place." What can the teacher do to help reach this climax?

SOLVING A PROBLEM is like crossing a busy street in traffic. There are many hazards. You may get across by sheer luck or by a series of skillful maneuvers. Either way you have a sense of exhilaration when you reach the other side.

A mathematics teacher never goes very far without defining his terms. So perhaps we had better agree on what we mean here by the word "problem." In algebra, when we say "problem," we are generally thinking of a "word-problem," so let us restrict ourselves here to that meaning of the word. By a "problem" in geometry we shall mean an "original," that is, an exercise to be proved, or a "construction," an exercise in which a figure of certain specifications is to be constructed by straightedge and compasses.

The essential elements of a problem are: (1) there is something you want, and (2) you don't know how to get it. There is inherent in every problem an element of difficulty, or of some obstacle to be overcome. It has been said that "education is honored when it is hard, and most honored when it is hard and good." A mathematics teacher could well apply that to problems —they are "hard and good." Our students might prefer to turn that around and call them "good and hard."

The greatest single ally we can have in teaching problems is interest on the part of the student. Interest might almost be put ahead of intelligence, although we can do with a little of both. Your experience will no doubt bear out the fact that with sufficient interest even the slow student can make some headway. This is better said in the words of Mark Van Doren: "There is no disciplinary value in a study that is not taught and learned with relish. When there is excitement in the learning, discipline becomes a privilege."

A problem has more likelihood of capturing the imagination of the student if it contains something that is new. A calculus student is excited when he meets maximum and minimum problems for the first time. But what about algebra problems in high school?

A short time ago on a television program Hume Cronyn and Jessica Tandy portrayed the parents of a small boy. They were attending a P.T.A. meeting at school. As they entered the school, the father looked around and said, "Gosh, what an old building!" The mother said, "Oh, this is the new wing. This part was built in a later administration—McKinley."

Many of the problems in our algebra textbooks are in that same state. A little improvement in this respect has been made in recent years. *A* and *B* who used to dig wells are now machines with a given output. The young men who got blisters on their hands rowing upstream and downstream are now piloting planes with and against the wind, a sensible situation in that it leads to wind-drift problems which the student will meet later in trigonometry—if he lasts that long. The

[1] Condensed from a paper presented at the annual meeting of The Association of Teachers of Mathematics in New England, at Boston University, December 4, 1954.

fact remains, however, that most of the problems we perpetrate on our students are either ancient or artificial. The problems that need to be solved in higher mathematics, in the sciences, in engineering, and in industry are too difficult for the secondary school student. Some time ago there was a short sketch in the "Accent on Living" section of the *Atlantic Monthly*, poking fun at the problems in algebra textbooks. The author of this sketch remarked that the problems built around things in real life are simply a hoax—that when peanuts at 33¢ a pound, walnuts at $87\frac{1}{2}$¢ a pound, and pecans at 72¢ a pound are mixed to get a mixture worth 69¢ a pound, the price is 69¢, but the mixture is nine-tenths peanuts, as everyone except an algebra student knows.

In view of the fictitious nature of the problems we ask the student to solve, we sometimes abandon a purely mathematical approach and resort to a psychological one. This is done in various ways. We spread the problems out—a few at a time over a longer period of time—knowing that assigning ten or fifteen problems a day for three or four consecutive days is a sure-fire formula for leaving a class behind us. We dramatize, coax, caper, cajole. Some teachers use toy trains or automobiles, or designate two points in the room as Boston and New York and have a train named Bob overtake a train named Bill. Let no teacher be ashamed to admit that he draws pictures of choppy waves and boats that look like bananas. It would be difficult for a college instructor who has not taught at the secondary school level to have any conception of the contortions —psychological, physical, dramatic, artistic, linguistic—that a high school teacher goes through in the process of exciting the interest of his students in solving problems.

To be able to do a problem in either algebra or geometry, a student must first know what the words of the problem mean. He cannot be expected to know, unless he is told, the meanings of mathematical words or of ordinary words used in a specialized sense. One way of introducing mathematical words as they occur is by considering word derivation. The meanings of words of Latin derivation can be made clear by taking the word apart to exhibit the root and the prefix. Most algebra and geometry students are taking or have taken Latin and can follow this dissection. Here are some examples.

tangent: *tangere*, 'touch'
secant: *secare*, 'cut' (bisect, trisect, intersect, sector, conic section)
quotient: *quot*, 'how many'
subtend: *tendere*, 'stretch,' *sub-*, 'under'
adjacent: *jacere*, 'lie,' *ad-*, 'at,' 'to'
exponent: *ponere*, 'place,' *ex-*, 'out'
opposite: *ponere*, 'place,' *ob-*, 'before'
transversal: *vertere*, 'turn,' *trans-*, 'across' (converse, inverse, vertex, vertical)
circumscribe: *scribere*, 'write,' *circum-*, 'around'
concurrent: *currere*, 'run,' *con-*, 'together'
consecutive: *sequi-*, 'follow,' *con-*, 'together'
commutative: *mutare*, 'change,' *com-*, 'together'
equivalent: *valere*, 'be worth,' *aequi-*, 'of equal'
contradiction: *dicere*, 'speak,' *contra-*, 'against'
interpolate: *polire*, 'polish,' *inter-*, 'between'
quadrilateral: *latus*, 'side,' *quadri-*, 'having four'
perpendicular: *pendere*, 'hang,' *per-*, 'thoroughly,' 'perfectly.'

Many common words have different meanings in different settings. It is not surprising that sometimes a student is mystified by a simple word used in a special sense. Perhaps you have had a student ask you what is meant by a "point without a circle." He can visualize "a man without an umbrella," but his imagination fails him at "a point without a circle." And sometimes a student has an erroneous conception of the meaning of a

word. Have you ever been startled to hear a student say, "I can do this problem by mathematics but not by algebra"?

To be able to handle problems effectively, the student must be retaught how to read. Reading, and particularly mathematical reading, is a discipline. E. B. White, discussing the future of reading in one of the informal essays in his book, *Second Tree from the Corner*, mentions "audio-visual aids, which ask no discipline of the mind," and contrasts them with reading, which, he says, "is the work of an alert mind, is demanding, and under ideal conditions produces finally a sort of ecstasy." Reading mathematics may be even more demanding than reading literature or history. Ordinary reading matter is diffuse. Often a whole paragraph contains but a single idea. Mathematical reading matter is condensed, concentrated—dehydrated, if you will; much significant information may be stored in a single sentence or even a single phrase. The mathematics student needs to be taught to change his reading habits. Ever since the student first learned how to read, teacher after teacher has been trying to teach him to increase his speed of reading. We must do the reverse. He must be taught to read slowly, to pause at commas or at the end of a single idea, to consider what he has read and decide what it means, to go back and re-examine it as many times as necessary. Bright students in particular need to be slowed up. When a bright student has difficulty with a problem, the reason may be that he has read the problem too fast. Also, bright students are accustomed to skimming. The ability to skim is an asset when used in the proper place and for the proper purpose. Skimming is out of place in reading mathematical problems.

It is a good practice to have the student attack a problem by re-stating it orally in his own words after he has read it. If he is able to state the problem in simple terms, he is well on his way. Thought and expression are so closely related that the student should be given every opportunity to develop facility in oral expression. (You probably know that a machine has been developed at Massachusetts Institute of Technology which produces an approximation to the human voice saying the three words, "How are you?" No one can hear this mechanical voice and be apprised of the research and expense that have gone into its production without having a deeper appreciation of the power of oral expression.) After a student has solved a problem, we should ask him to explain it, to give a clear oral explanation. Giving an oral analysis of the completed problem may help him to solve other problems, because it requires him to re-examine the path by which he has come.

Have you ever pondered the fact that the College Board Scholastic Aptitude Examination has two parts and only two—*verbal* and *mathematical*? Language and mathematics are basic fundamental equipment of a literate human being. I would like to quote in this connection a remark from the *Notebooks* of Robert Frost. He says, "School is founded on the invention of letters and numbers. The inscription over every school door should be the rhyme ABC and One Two Three. The rest of education is apprenticeship and for me doesn't belong in school." And it has been said elsewhere that language and mathematics are the mother tongues of the human race, and no student, whatever his talents or plans for the future, should be permitted to remain speechless in either tongue.

Newton compared the setting up of an equation in problem-solving to translation from one language to another, and probably every algebra teacher uses the word "translate" at one time or another. Stuart Chase describes algebra as "direct shorthand translation of longhand talk, a tidier, defter language." Slang, he notes, sometimes performs a similar service. The American policeman does not say, "Disperse yourselves as rapidly as possible." He says, "Scram!"

To be able to translate from one language to another, a person must understand what is said in the first language and the vocabulary, syntax, and idiomatic structure of the second. Now it is common experience that it is harder to translate into a foreign language than from it—German composition is more difficult than German translation. Why, then, don't we give more practice in translating from algebra into words as a preliminary to translating words into algebra? As an example of this, consider the statement

$$8-3=5.$$

How many ways can the class find to express this statement in words?

1. Eight minus three equals five.
2. Eight less three is five.
3. If you subtract three from eight, you get five.
4. Eight diminished by three gives five.
5. Eight exceeds three by five.

And perhaps there are other ways. Practice of this kind could also be worked in connection with signed numbers, as in the statement

$$(+7)-(-2)=+9$$

and finally the transition could be made to simple equations. The statement,

$$2x+3=11,$$

could be translated to: "I am thinking of a number. If I multiply it by 2 and add 3, I get 11. What is the number?" or it could be worded in other ways. Proceeding gradually from simple to more complicated equations, the class might eventually be led to formulate in their own words such equations as:

1. $.60(15)+.90x=.75(15+x)$
2. $.06x+.05(x+6000)=.0525(2x+6000)$
3. $\frac{1}{8}+\frac{1}{12}=\frac{1}{x}$ or $\frac{x}{8}+\frac{x}{12}=1$
4. $\frac{1}{18}+\frac{1}{24}-\frac{1}{15}=\frac{1}{x}$ or $\frac{x}{18}+\frac{x}{24}-\frac{x}{15}=1$
5. $\frac{75-x}{30}+\frac{x}{22}=2\frac{38}{60}$
6. $\frac{43}{12-x}=\frac{26\frac{1}{2}}{12+x}+2$
7. $\frac{x}{4\frac{1}{2}+1\frac{1}{2}}+\frac{x}{4\frac{1}{2}-1\frac{1}{2}}=6$

Equations like 6 and 7 may suggest to you a line of Ogden Nash's: "The gamey fish swim upstream, but the sensible fish swim down."

In translating in the reverse order, from words to algebra, the usual practice is to start with problems that are directly translatable by grouping the words. One device is to draw a vertical line down the middle of the page and write the word phrases on the left, one to a line, and their algebraic equivalents on the right. Prerequisite for this, of course, is the ability to express phrases about quantities as algebraic expressions, as in digit-problems $10t+u$, etc. Much preliminary work of this nature is necessary, as every algebra teacher knows.

In more complicated problems intermediate steps, such as making a table or drawing a diagram, may help in getting from the sentence to the equation. A diagram is useful in all problems involving geometric figures, as well as in motion and lever problems. If the student uses a table, or "box," as we sometimes call it, it would be well for him to write above it or beside it the formula he needs to use in filling it out, such as $rt=d$, $pr=i$, $lw=A$, and the like. Some students fill out the box successfully and still can't write the equation. They crawl into the box, pull down the cover, and can't get out again. To reduce the number of box-casualties it may help to ask, "Is there any additional information given besides that which you have used to make out the box?" It may be that they haven't yet used the fact that the total amount of interest is $345 or that the total time for the trip is $7\frac{1}{2}$ hours. However, the extra fact may be only implied, not explicitly stated. Perhaps you

saw in a recent issue of *The Saturday Review* a cartoon of an ultramodern house. This house was made entirely of glass, except the windows, which were made of brick. Now problems, like this cartoon, often contain unstated implications. For example, "It takes 3 hours for one train to travel a distance that another train can travel in 2 hours." It is a rare student who at his first reading of such a statement grasps the fact that "a distance" really involves two expressions that are equal. Other situations in which there are subtle equalities are: (1) When two trains approach each other from opposite directions, the sum of their distances when they pass each other is equal to the distance between their starting-points. (2) If two trains start from the same place in the same direction, one earlier, one later, their distances are equal at the moment that the faster train overtakes the slower. (3) Whether A does a piece of work in 6 hours or in 5 hours, the whole piece of work is 6/6, or 5/5, or 1. (4) If you add water to an alcohol mixture, the amount of alcohol you end with is the same as the amount of alcohol you started with. On the other hand, there may be a formula hidden in the problem, such as the theorem of Pythagoras in a problem involving the diagonal of a rectangle.

There are several other things we teachers can do. A few of them are:

1. Encourage the student to estimate the size of the answer before he begins. "If A can do a piece of work in 8 days and B can do it in 7 days, how long will it take them to do it working together?" Well, if both worked at A's rate, they could do in 4 days working together. If both worked at B's rate, it would take them $3\frac{1}{2}$ days. So the answer will be between $3\frac{1}{2}$ and 4 days.

2. This suggestion is a "don't." Don't do the problem for the student; let him poke around in it himself. We teachers may give him hints and clues, ask him leading questions, but we shouldn't do the problem for him. Don't you think someone should write a song with the title "I Can See It When *You* Do It, But When I *Get Home*—"?

3. The student should be led to acquire the habit of reading the problem again as soon as he has arrived at the value of x. Perhaps the answer to the problem is not x, but $3x$ or $4x$, as is often the case in problems involving ratios.

4. Has he labelled the answers correctly? In an intermediate algebra class a short time ago, the following problem occurred:

> "When \$6700 is invested, part at 4% and the remainder at 5%, the yearly income is \$6 greater than if the entire amount had been invested at $4\frac{1}{2}$%. Find the amount invested at each rate."

The girl who explained the problem had for her equation

$$.04x+.05(6700-x)+6=.045(6700).$$

Her answers were $x=\$3950$ and $\$6700-x=\$2750$, unlabelled. No one in the class protested that the equation was incorrect. On investigation it was found that some members of the class disagreed with her equation, but they had arrived at the same answers, so they let it go at that. A great deal of heated argument followed. When an agreement was finally reached on the correct equation, the class started over again and found that while the incorrect equation had produced the answers \$3950 at 4% and \$2750 at 5%, the correct equation gave just the reverse, \$2750 at 4% and \$3950 at 5%. Someone asked if you would always get the answers reversed if you put the \$6 on the wrong side of the equation, or if it was just because it was \$6700. The problem was then tried with k dollars instead of \$6700, using the equations:

(1) $.04x+.05(k-x)+6=.045(k)$

(2) $.04x+.05(k-x)-6=.045(k)$.

The roots were:

$$(1)\begin{cases} x=\dfrac{k}{2}+600 \\ k-x=\dfrac{k}{2}-600 \end{cases} \qquad (2)\begin{cases} x=\dfrac{k}{2}-600 \\ k-x=\dfrac{k}{2}+600 \end{cases}$$

so that the same situation prevailed. Finally the question arose, did it have to be \$6, or would it still be true for some other number of dollars? This time a generalization was made of the rates and the \$6, as well as of the total investment. The equations now read:

$$(3)\quad \frac{a}{100}x+\frac{b}{100}(k-x)+m=\frac{c}{100}(k)$$

$$(4)\quad \frac{a}{100}x+\frac{b}{100}(k-x)-m=\frac{c}{100}(k)$$

The roots were now:

$$(3)\quad \begin{cases} x=\dfrac{ck-bk-100m}{a-b} \\ k-x=\dfrac{ak-ck+100m}{a-b} \end{cases}$$

$$(4)\quad \begin{cases} x=\dfrac{ck-bk+100m}{a-b} \\ k-x=\dfrac{ak-ck-100m}{a-b} \end{cases}$$

By this time the class was in pretty deep, and some of them were tempted to abandon the search, but with a little encouragement they proceeded to determine the conditions under which the solutions of (3) are the solutions of (4) in reverse. Equating x of (3) to $k-x$ of (4) and x of (4) to $k-x$ of (3), they found that

$$c=\frac{a+b}{2}$$

that is, that the rate on the entire amount is the arithmetic mean of the rates on the separate investments, a condition which they noted was fulfilled by the 4%, $4\frac{1}{2}$%, and 5% of the original problem. Thus far it had been proved only that the condition was necessary. With a little prompting the better students were able to establish the sufficiency of the condition. More real interest was stirred up by that problem than by any other single item all year. That problem was a "natural" for teaching three things: the desirability of labelling answers, the meaning of "necessary and sufficient" as used in mathematics, and the power of generalization.

5. It is good to surprise a class occasionally with a problem that results in an indeterminate equation or a problem whose solution is impossible under the conditions given. Is there anything to match the gleam in the eye of the militant student who marches up to your desk at 8 A.M. to tell you that either you or the book has made a mistake?

Problems in geometry are often called "originals." That is a good name for them—they take originality, imagination, ingenuity. One of the reasons students have difficulty with originals is that the proofs which they encounter in their textbooks are synthetic proofs. A synthetic proof does not show how the proof was discovered. The proof the student reads is the result of a long arduous process of analysis, but the analysis is not recorded. All he sees is the finished structure—the scaffolding has been taken down.

There is general agreement that there are three steps in the process of proving originals: (1) analyzing the hypothesis; (2) analyzing the conclusion; (3) finding the connection between the hypothesis and the conclusion. A fourth step universally recommended is a thorough discussion of the completed problem in all its aspects, including generalization if possible, or, in the case of constructions, considering the conditions under which the construction is possible.

These three steps in proving an original can be paraphrased in words of one syllable: (1) What have you got? (2) What do you want? (3) How can you use what you've got to get what you want? (Parenthetically, don't these summarize practically all human endeavor?)

Steps (1) and (2) may be discussed together. First, the student must analyze and organize his data. To analyze means to separate a thing into the parts or elements of which it is composed. He might first make a free-hand sketch to guide him,

then draw the figure carefully. A well-drawn figure will sometimes indicate the line of proof to be followed. Next he should analyze and organize his conclusion. He must be able to state clearly all the facts about the figure which are given and all the facts which he is to prove. The parts of the figure should be labelled by a suitable symbolism. In fact, with a carefully chosen symbolism it is possible to carry through a whole proof symbolically.

Step (3) is the crux of the matter. How is the student to bridge the gap between hypothesis and conclusion? How is he to get from Here to There? The connection is to be established by means of a series of connecting links carefully selected from the entire body of geometry he has had up to this time. This body of materials consists of undefined terms (*point, line, plane*), defined terms (like *midpoint, complementary angles, perpendicular-bisector*), assumptions, and previously proved theorems and corollaries. It is evident that he must know all the geometry that has gone before.

The student may now consider each part of the hypothesis separately and ask himself or be asked by the teacher what consequences follow from that given fact, and will any of them help him in progressing to the desired conclusion? He may also consider each part of the conclusion and determine the different antecedents which would lead to such a conclusion, and ask whether any of them are available. Some teachers have students keep an expandable notebook list of all the consequences which follow from a given fact, such as all the things he knows to date about a parallelogram, and also a list of all the ways he knows of proving a specified relationship, such as the various ways of proving two lines perpendicular. In one form or another he needs to have these things handy. The plumber must not come to the job without his tools.

But having the tools is not enough. He must be able to select the ones relevant to the particular situation at hand and suitable for his purposes, and he must know how to use them. He needs to explore all the possibilities at his disposal, combining all his previous knowledge and experience with all the aspects of the new situation until a method emerges. If he comes to a dead end by one approach, he must back up and make a detour. He may find the need of drawing auxiliary lines in the figure and starting all over again. During this period of probing, the teacher can help him by adroit questioning. There would be less wear and tear on the teacher if he were to show the student the connection, but in the long pull (and it is a long pull) the student will be much better off if he is led to find it out for himself.

If the connection is not soon manifest, let him try the "analytic method of proof." The analytic method, which is also applicable to constructions, consists of a series of retrogressive steps, starting with the conclusion A. He finds that he can prove A if he can prove B. He can prove B if he can prove C, and so on. This sequence is pursued until a step is reached which is known to be true, then the process is reversed. Polya, in his book *How to Solve It*, compares the analytic method with the problem of a primitive man who wishes to cross a creek. The crossing becomes the object of a problem, the x of this primitive problem. He recalls that he has crossed some other creek by walking along a fallen tree. He looks around for a fallen tree, which becomes his new unknown, his y. He cannot find a suitable fallen tree but there are plenty of trees standing along the creek. Could he make a tree fall across the creek? There is a new unknown: by what means could he tilt the tree over the creek? This train of ideas is analysis. The synthesis is the translating of ideas into actions to bring about the desired result. Walking across the creek, which was the first desire from which the analysis started, is the last act with which the synthesis ends.

Before the student will be able to apply the analytic method of proof to his own

problems, he needs to have seen it applied by the teacher to many of the textbook theorems. Developing the textbook theorems by leading the student to discover the proof analytically before letting him read the synthetic proof in the book is excellent preparation for proving originals.

Throughout this discussion of originals, our attention has been directed to the student who has difficulty with originals. There are some students for whom originals are play. Such a student likes to explore territory beyond the required work. There are many areas that are fertile ground for his activity, for example the multi-converse concept. Examining all the converses of a given theorem gives him not only a chance to state theorems himself but the added pleasure of determining whether each of them is true, proving it if it is true, or producing an exception to show it false. This student may even be given originals without conclusion: what do you think follows from this hypothesis, what is the strongest conclusion you could prove? This gives his intuition a workout, and intuition needs exercise as much as reason. A mathematics theorem at any level is a hunch before it is a theorem. The world has a great need for people who can make intelligent inferences.

The psychology of creation enters into the solving of a problem. Solving a problem is invention, and invention has no set of rules. If you are interested in the creative element in problem-solving, read "Mathematics and the Arts," by Marston Morse, in the *Yale Review*, June, 1951. In solving a problem a student becomes, in a sense, a creative artist and experiences the joys and sorrows attendant upon the process of creation. The climax of the problem-solving process occurs at the moment when all the pieces suddenly fall into place and he can say, "I've got it!" That moment of illumination is the apex of the creative process. Analysis precedes it; synthesis follows it.

Mathematics does not have the monopoly on problem-solving; problems are a universal phenomenon. Wherever there are human beings, there are problems. The research scientist who is seeking to identify the unknown in cancer is working on a problem, a big one. Problems in any and all fields have this in common: solving them is a progression from given data to desired goal by studying both data and goal, conjecturing a possible connection between them, and selecting the proper means for establishing that connection. Human nature finds one of its greatest satisfactions in solving problems.

"Because of the rather special and intense motivation in the student to acquire skills useful in his own field, the method of teaching is critical. One simply cannot teach mathematics to social scientists of today the way he teaches it to mathematics students; the motivation, the system of needs, the goals of these two kinds of students are entirely different. I think the social science student of mathematics must have frequent reinforcement from manipulative skill at working easy problems—problems he previously could not handle. Such reinforcement will maintain his motivation."—*R. R. Bush, "Mathematics for Social Scientists,"* American Mathematical Monthly, *October 1954*

Watch your figures

MAUDE SEXTON, *Alexander Hamilton Junior High School, Seattle, Washington.*

Displaying an attractive bulletin board is a good way to attract attention and stimulate interest.

THE REQUEST to sponsor a large school bulletin board presented a fine opportunity to my five junior high school mathematics classes. The children welcomed the idea of displaying their work in mathematics in original and attractive form for the rest of the school to enjoy. It was agreed that every pupil would have the privilege of preparing one problem for the display that would be typical of the class work being done. Each paper was to show pictorially the problem and the steps of solution. Cartoons were the most popular illustrative devices used by the pupils, although a few used actual pictures to illustrate their problems.

One seventh-grade class decided to create a display which would compare the three patterns of percentage. The other two classes of seventh-graders were to use materials from their science class. They planned to build circle graphs which would use their newly found "playthings," compasses and protractors. Each youngster chose a different food, changing its food value percentages to degrees, using common or decimal fractions. Contributions of "story problems" were made by members of two ninth-grade algebra classes. Problems were chosen to show practical uses of algebra. The illustrations were worked out and the steps detailed in a fashion to provide a sort of crystal ball into which student sight-seers could look for the possible answers to the mysteries of algebra. These algebraic collections included blends of coffee, mixtures of nuts, dividing a job fairly, crew races, bicycle trip time schedules, filling a water tank, a post measured above and below the ground level, and the division of some coins. No two problems were alike.

An atmosphere of freedom and friendliness soon developed as the youngsters' imaginations were free to work. Since space was limited, it was necessary for each class to decide which papers were to be displayed. As soon as his problem was completed, each pupil whose paper was chosen took care of mounting his own work attractively on colored paper with the help of his art, mechanical drawing, or printing teacher. On the appointed day each child thumbtacked his contribution into place on the bulletin board. The title for the bulletin board, "Watch Your Figures," was decided upon by all of the classes. Two ninth-grade girls volunteered to cut out the letters for the title and mount them.

After the bulletin board was complete it attracted much valuable comment from the entire school—both students and faculty. Some of the seventh-graders remarked: "When I go past the third-floor bulletin board and see people stop to look at it, I feel that all the work I put on my paper was worth it." "It makes me proud of the room, even though I didn't have a piece of work up" and "I really got a reward in just seeing the nice effect it had on the board and the good work the kids really put into it." The girls say, "It's lovely," and the boys remark, "It's real

neat!" Some of my ninth-graders commented: "I did not think at first I would be able to explain and express myself, but in doing this work I have gained self-confidence." "It showed me how to start story problems, and I believe one reason it was so effective was that it showed each step." "I realize how much easier it is to work story problems if you can first start them in picture form." And, "By looking at the work of others, you can see where you make mistakes."

"The moral . . . seems to be that mathematical reasoning based on an axiomatic basis is not difficult, and the traditional British preference for algebraic methods is merely the result of the intensive drilling in algebra and algebraic methods which our students undergo. Modern mathematics is turning more and more to axiomatic methods, and there is a great deal to be said for introducing students to them at an early stage, when, I believe, they would take to them more readily, and enjoy them."—*Taken from the presidential address of W. V. D. Hodge, on "Changing Views of Geometry" delivered at a meeting of The Mathematical Association (British).*

For a better mathematics program[1] *1) In the college*

E. H. C. HILDEBRANDT, *Northwestern University, Evanston, Illinois.*

Colleges, too, are confronted with the problems of providing a more adequate program in mathematics. However, as Felix Klein points out, the colleges face psychological problems which force them "to harmonize contradictory requirements."

Introduction

The basic assumptions underlying this article are that the teaching of mathematics in our colleges can and should be improved, and that a careful consideration of some of these areas-for-improvement will not only prove of benefit to us as individuals, but also to the progress of mathematics teaching in our colleges. Our discussion will be limited to the mathematics usually offered in the first two college years and will not go beyond the full year's course in the calculus.

Our attention will be drawn to ten areas that are not necessarily independent of one another. Some sections deal with the improvement of teaching as it affects the student's own needs and interests such as his mathematical maturity, his attitude toward mathematics, what benefits may result from his study of mathematics, and how learning in mathematics takes place. The remaining parts deal with methods and materials of instruction, curriculum revision, influence of the textbook on teaching, testing, and the training of teachers.

[1] The three articles under the general heading "For a Better Mathematics Program" were delivered as addresses at the Fifteenth Summer Meeting of The National Council of Teachers of Mathematics at Bloomington, Illinois, August 23, 1955.

The place of mathematics in college education

Some years ago a popular story among college students concerned the bookstore that advertised a second-hand copy of Love's *Calculus* on its bargain blackboard. This announcement caught the eye of one of the college wags, who in an undetected moment added the word "Who?" in front of the listing. Perhaps today we may shrug off such a remark as, "Who loves calculus?" by replying, "Who doesn't?" if we happen to be teaching in a college where all of our students do like their calculus courses.

Suppose, however, that there are some schools in which students do not like mathematics. How popular is mathematics and does it at least hold its own in comparison to other subject matter departments? One way to find out would be to request our students to list the courses in which they have been enrolled during the semester or year and to rank them from best to least liked. Assuming that students take five three-hour courses, we should expect to find mathematics at the top of the list in at least 20 per cent of the cases. In one junior high school such information was collected for four consecutive years, and the following per cents on favorite subjects were tabulated:

	1949	1950	1951	1952
Mathematics	38	$43\frac{1}{2}$	32	$13\frac{1}{2}$
Social Studies	32	8	27	51
English	13	$6\frac{1}{2}$	11	6
Science	7	$10\frac{1}{2}$	7	10
Others	10	$31\frac{1}{2}$	23	$19\frac{1}{2}$

In this case, mathematics stayed at the top for three consecutive years. The fourth year, a new teacher in mathematics was assigned to the school. Joseph Seidlin,[2] in a study published in 1931, reported that in one college, out of 375 students in freshman mathematics, only 12 continued with mathematics the next year, and that students who did excellent work in high school mathematics lost interest in courses in college mathematics or dropped out of them.

A number of colleges have tried course evaluations in some or all of their courses. Questionnaires are usually distributed to students near the close of the course, are returned unsigned, and placed in an envelope that is sealed in class. It is understood that the returns are not to be examined until the instructor has handed in his final reports of grades. The student's appraisal is made on an A, B, C, D, F rating scale for items such as: organization of the course, how course has stimulated his interest in the subject, how it has encouraged independent thought, how well the lectures have been organized, the relevance of examinations to material covered in the course, and the clarity of examination questions. Additional information requested may include: reactions to the difficulty of problem and reading material, the average number of hours spent per week in outside preparation, the purpose of the course as he saw it, the strong points of the course, and suggestions for improving the course. Some mathematics instructors have expressed the opinion that freshman and sophomore college students are incapable of passing satisfactory judgment in items such as the above. Others state that they have profited from such responses.

While the above approach may prove of benefit to the individual instructor, what we really need is a well-conducted survey made by a national organization. Business and industry use such polls to advantage. Why should not we?

Aims of College Mathematics

If the average instructor were asked to list the aims of mathematical instruction, he might give the same answer as did Professor G. A. Miller, famous not only for his mathematical writings and his many years of teaching at the University of Illinois, but also for his ability to accumulate a fortune by a wise choice of investments. About thirty-five years ago he wrote:

"The principal aim of college mathematics is not the training of the mind but the providing of information which is absolutely necessary to those who seek to work most efficiently along various scientific lines. Mathematical knowledge rather than mathematical discipline is the main modern objective in the college courses in mathematics."[3]

Others feel, however, that our aims are much broader than this. For example, the report of the National Committee on Mathematical Requirements (prepared by such men as A. R. Crathorne of the University of Illinois, C. N. Moore of the University of Cincinnati, E. H. Moore of the University of Chicago, D. E. Smith of Columbia University, and H. W. Tyler of Massachusetts Institute of Technology) indicated the importance of such goals as:

(1) The acquisition, in precise form, of those *ideas or concepts in terms of which the quantitative thinking of the world is done.* . . .

(2) The development of *ability to think clearly in terms of such ideas and concepts.* . . .

(3) The acquisition of *mental habits and attitudes* which will make the above training

[2] Joseph Seidlin, *A Critical Study of the Teaching of Elementary College Mathematics* (New York: Bureau of Publications, Teachers College, Columbia University, 1931), p. 2.

[3] G. A. Miller, "The Teaching of Mathematics," *College Teaching*, ed. Paul Klapper (New York: World Book Company, 1920), p. 172.

effective in the life of the individual. Among such habitual reactions are the following: a seeking for relations and their precise expression; an attitude of enquiry; a desire to understand, to get to the bottom of a situation; concentration and persistence; a love for precision, accuracy, thoroughness, and clearness, and a distaste for vagueness and incompleteness; a desire for orderly and logical organization as an aid to understanding and memory. . . .

(4) The development or acquisition of *appreciation of the power of mathematics*—of what Byron called "the power of thought, the magic of the mind" and the role that mathematics and abstract thinking, in general, have played in the development of civilization; in particular in science, in industry, and in philosophy.[4]

Professor Bôcher of Harvard University referred to some of these same aims in his paper before the Department of Mathematics of the International Congress of Arts and Sciences held in St. Louis in 1904 when he said:

I like to look at mathematics almost more as an art than as a science; for the activity of the mathematician, constantly creating as he is, guided though not controlled by the external world of the senses, bears a resemblance, not fanciful I believe but real, to the activity of an artist, of a painter let us say. Rigorous deductive reasoning on the part of the mathematician may be likened here to technical skill in drawing on the part of the painter. Just as no one can become a good painter without a certain amount of this skill, so no one can become a mathematician without the power to reason up to a certain point. Yet these qualities, fundamental though they are, do not make a painter or a mathematician worthy of the name, nor indeed are they the most important factors in the case. Other qualities of a far more subtle sort, chief among which in both cases is imagination, go to the making of a good artist or good mathematician.[5]

It should be a matter of deep concern to every mathematics teacher whether his courses aim only at "providing information" that is useful "along various scientific lines" or whether they can serve even wider aims than those stated by the National Committee. If the latter is the case, then mathematics may continue in the role of "queen and handmaiden of the sciences," but if our aims go no further than "providing information [useful] along various scientific lines" then our subject is relegated to the menial position of "the slave of the sciences."

The mathematics curriculum

The past forty years have seen considerable experimentation with revisions in and development of materials for our first-year college courses. Some colleges report success in providing a one-year unified course, which includes the important topics in trigonometry, college algebra, and analytic geometry, and which introduces some of the concepts of the calculus. Other schools have attempted to develop a one-year general cultural course for students not necessarily planning to major in mathematics with the hope that some might find the field so fascinating that they would decide to continue with the conventional courses. A number of colleges and universities are making serious attempts to work out a program whereby a student can complete a study of the differential and integral calculus by the end of his freshman year. Many teachers will be interested in the descriptions of the experimental programs at the University of Chicago, Yale University, and the University of Kansas, which were described at the recent Summer Institute for Teachers of Collegiate and High School Mathematics sponsored by the National Science Foundation and held at the Oklahoma Agricultural and Mechanical College in the summer of 1955.[6]

Unfortunately, very little proof indicating the actual success which these experimental programs have attained has been published or made available. Actual data comparing the progress and success that students have attained in these courses compared with students who have taken

[4] National Committee on Mathematical Requirements, *The Reorganization of Mathematics in Secondary Education* (Mathematical Association of America, 1923), pp. 9–10.

[5] M. Bôcher, "The Fundamental Conceptions and Methods of Mathematics," *Bulletin of the American Mathematical Society*, XI (December 1904), p. 133.

[6] Saunders MacLane, E. G. Begle, and G. Baley Price, *Lectures on Experimental Programs in Collegiate Mathematics* (Stillwater, Oklahoma: Oklahoma A & M College Bookstore, 1955).

the conventional courses is needed. Proof should be based on standard examinations. In addition, data showing a commendable increase in the number of students electing advanced courses in mathematics could do much to influence other schools to make similar improvements in their programs.

Mathematical maturity of college students

How mature is the mathematical thinking of our student when he begins his college work? Is he able to cope with the concepts of the calculus and of some modern mathematics in his freshman year? These are some of the questions that go unanswered year after year. As a result, many schools have a high mortality in their freshman courses during the first week and continuing through the entire semester.

What is needed is a definite testing program, which would determine the capabilities and the weaknesses of the individual student in mathematics and in areas that could be related to success or failure in the study of mathematics. Such tests given during the first weeks of the year should not only show the skill or lack of skill in handling specific basic concepts of arithmetic and algebra, but should also supply information on such factors as the ability to make correct observations, to handle data or information correctly, to draw correct conclusions, and to reason properly. Certain standards in mathematics for entrance into our freshman courses should also be agreed upon on a national basis. High schools would gladly cooperate if our standards were specifically indicated.

In addition, an introduction to the basic concepts of the usual college mathematics could be presented in such a way as to arouse the student's curiosity and interest. At the same time, continued testing could indicate what progress the student is making in overcoming previously discovered weaknesses. By the end of the semester, one should really have sufficient information to determine whether the student has possibilities of succeeding in the field. Such a plan would involve more attention to our students in their freshman year than we may be giving them now, but in the face of increasing needs for mathematically trained individuals a program with such a purpose would certainly be worth the effort.

Methods of instruction

The most common method of instruction is no doubt the recitation method. Seidlin reported that 85 per cent of the classes he observed were conducted by this method. He recorded 150 observations of freshman and sophomore courses in 19 eastern colleges taught by experienced teachers. These observations were divided into seven groups as follows (the numbers in parentheses indicate the number of cases in each group):[7]

(1) The lecture which goes far afield, generally instructive and entertaining. The students, however interested, remain distinctly passive. (8)

(2) The lecture used as a method of developing a theory, or a problem, or an exercise, occasionally interrupted either by the instructor's question or by a student's inquiry. Although the students do not say much, they nevertheless seem to participate "actively" in the proceedings. (14)

(3) The recitation at which the time is fairly evenly divided between instructor and students. The students' questions appear to be spontaneous. The instructor is resourceful and enthusiastic. (10)

(4) A combination of blackboard and oral recitation. The students show a lively interest; the instruction is casual and entertaining. (8)

(5) A ground-covering, textbook repeating recitation, at which students participate technically and mechanically. (45)

(6) A carefully planned question and answer development. (4)

(7) A blackboard recitation, at which the students recite and get a grade. (61)

The following characteristics of the "ground-covering textbook repeating recitation" (No. 5) as Seidlin records them, may be of interest:[8]

[7] Seidlin, *op. cit.*, pp. 1–2.
[8] *Ibid.*, p. 43.

(1) They "cover the ground": Most of the recitation is devoted to the content and method of "so many pages" of the textbook.

(2) They generally lack motivation. Everything is done apparently for no other reason than that of "satisfying requirements."

(3) They are quite matter of fact. Nothing happens to relieve the monotony of a formal recitation.

(4) Thought organization is readily sacrificed. If some topic is inadvertently omitted, it is brought in later regardless of the fitness of things.

(5) Students' questions are rarely genuine. Such questions as are asked obviously aim at the next test. As for example, "How do we know when to solve by formula and when to solve by completing the square?"

The blackboard recitations (No. 7) seemed to conform to the following pattern:[9]

The instructor assigns (by reading from the textbook) individual examples to twelve students at the blackboard, and one exercise to the remaining eleven students in their seats. After twenty-five minutes, the students who were at the blackboard are called upon to recite on their own work. Save for corrections of errors, there are practically no interruptions. As a student's name, or number of the board, is called, he rises and either at his seat or at the blackboard recites what is already in white on black. Just as soon as he has finished, another takes his place. Thus nine students repeat their work orally. The tenth is interrupted by the bell and the assignment for the next period. The instructor marks each student at the end of the student's second appearance. He collects the papers of the others, and also records on the basis of the written evidence the marks of the three students who did not recite.

Instructors in these 150 classes asked 3126 questions, while only 481 inquiries were made by students. The average number of questions asked by instructors for the classes assigned to the seven groups were as follows:[10] "(1) 1; (2) 2; (3) 30; (4) 23; (5) 30; (6) 36; (7) 18."

The reason we have indicated this amount of detail from Seidlin's study is because the same tendencies seem to persist in mathematics classes today. For example, one well-known university professor speaking informally recently said:

Young teachers talk too much themselves and do not have much student participation. This is due to the fact that they have recently been exposed to lectures as a method of instruction and so they do not have sense enough to change their method when they meet up with freshmen who are just out of high school and who really do not know how to absorb a lecture. The other extreme is the instructor who follows the kind of teaching he had as an undergraduate which involved putting problems on the board which the students have worked out already and then going through the dry "rigmarole" of explaining what he has on the board, which bores every one excepting the particular student.

One of the reasons why instructors give little attention to methods of instruction is often attributed to their greater concern for research. Yet such mathematicians as Gauss, Boole, Babbage, Klein, Bôcher, and Bell made numerous suggestions which could well become a part of a teaching manual for every college teacher. For example, Gauss in his inaugural lecture to a class in astronomy at the University of Göttingen in 1808 points out that while oral instruction may be used in the beginning of the course to illustrate correct ways of thinking, it can at a certain point in the course actually harm the student:[11]

Oral instruction is merely to serve as a guide in the first steps, is to prevent erroneous views from taking firm root in the beginning and to prevent one from attempting incorrect and irrelevant ways, is to give opportunity for exercising the mathematical imagination faculty and one's own reflective meditation, in a word, oral instruction is to lead to the point from which one can progress further through independent study. On a certain point in the mathematical sciences oral instruction is not merely superfluous, but even injurious: one is not compelled to have a fault-finder from the beginning to the end in his course, but with time "comes of age"; certainly it is a hundred times more valuable if one solves a difficulty through independent exertion, than if one always needs correction from someone else, just as successful results of one's own meditation are always a hundred times more valuable than all borrowed wisdom. To be sure one must, on the other hand, not assume too early that one is exempt from the necessity of oral instruction: as long as one still

[9] *Ibid.*, p. 52.
[10] *Ibid.*, p. 82.

[11] Waldo Dunnington (ed. and trans.), *Carl Friedrich Gauss* (Baton Rouge: Louisiana State University Press, 1937), pp. 52–53.

runs against a snag and finds difficulties in the study of writings which are not notoriously faulty, obscure, and indistinctly written, oral instruction is still indispensable if one cannot resolve the difficulties.

Felix Klein was concerned about teaching abstract and difficult concepts too early in the course:[12]

From the standpoint of mathematical pedagogy, we must of course protest against putting abstract and difficult things before the pupils too early. In order to give precise expression to my own point of view on this point, I should like to bring forward the biogenetic fundamental law, according to which the individual in his development goes through, in an abridged series, all the stages in the development of the species. Such thoughts have become today part and parcel of the general culture of everybody. Now, I think that instruction in mathematics, as well as in everything else, should follow this law, at least in general. Taking into account the native ability of youth, instruction should guide it slowly to higher things, and finally to abstract formulations; and in doing this it should follow the same road along which the human race has striven from its naïve original state to higher forms of knowledge. It is necessary to formulate this principle frequently, for there are always people who, after the fashion of the medieval scholastics, begin their instruction with the most general ideas, defending this method as the "only scientific one." And yet this justification is based on anything but truth. To instruct scientifically can only mean to induce the person to think scientifically, but by no means to confront him, from the beginning, with cold, scientifically polished systematics.

Klein also points out that more attention should be given to the psychological rather than to the logical development:[13]

In Weber-Wellstein, the entire structure of elementary mathematics is built up systematically and logically in the mature language of the advanced student. No account is taken of how these things actually may come up in school instruction. The presentation in the schools, however, should be psychological and not systematic. The teacher, so to speak, must be a diplomat. He must take account of the psychic processes in the boy in order to grip his interest; and he will succeed only if he presents things in a form intuitively comprehensible. A more abstract presentation will be possible only in the upper classes. For example: the child cannot possibly understand if numbers are explained axiomatically as abstract things devoid of content, with which one can operate according to formal rules. On the contrary, he associates numbers with concrete images. They are numbers of nuts, apples, and other good things, and in the beginning they can be and should be put before him only in such tangible form. While this goes without saying, one should—*mutatis mutandis*—take it to heart, that in all instruction, even in the university, mathematics should be associated with everything that is seriously interesting to the pupil at that particular stage of his development and that can in any way be brought into relation with mathematics.

Further in his Evanston Colloquium Lectures on Mathematics at the time of the first World's Fair in Chicago in 1893, Klein recognized the problem of the instructor who feels duty bound to develop all the refinements of modern mathematics, but realizes that his students are not mathematically mature enough to grasp such a development:[14]

Now, just here a practical difficulty presents itself in the teaching of mathematics, let us say of the elements of the differential and integral calculus. The teacher is confronted with the problem of harmonizing two opposite and almost contradictory requirements. On the one hand, he has to consider the limited and as yet undeveloped intellectual grasp of his students and the fact that most of them study mathematics mainly with a view to the practical applications; on the other, his conscientiousness as a teacher and man of science would seem to compel him to detract in nowise from perfect mathematical rigour and therefore to introduce from the beginning all the refinements and niceties of modern abstract mathematics. In recent years the university instruction, at least in Europe, has been tending more and more in the latter direction; and the same tendencies will necessarily manifest themselves in this country in the course of time.

It is my opinion that in teaching it is not only admissible but absolutely necessary to be less abstract at the start, to have constant regard to the applications and to refer to the refinements only gradually as the student becomes able to understand them. This is, of course, nothing but a universal pedagogical principle to be observed in all mathematical instruction.

[12] Felix Klein, *Elementary Mathematics from an Advanced Standpoint* (New York: Macmillan Co., 1932; Dover Publications, Inc., 1954), p. 268.

[13] *Ibid.*, p. 4.

[14] Felix Klein, Evanston Colloquium Lectures on Mathematics, September 2, 1893, "On the Mathematical Character of Space Intuition and the Relation of Pure Mathematics to the Applied Sciences" (New York: Macmillan Co., 1894), pp. 49–50.

Likewise Eric T. Bell is quite emphatic in indicating his disapproval of too rigorous a course in calculus for engineers and mathematical physicists:[15]

> It may be emphasized once for all that a close description of how mathematics handles the problem of continuous change is neither possible nor desirable in a first approach. Further, to turn aside for a moment, if I may be allowed to vent a heretical opinion, I strongly disbelieve in ever giving either engineers or mathematical physicists a rigorous course in the calculus—the mathematics of continuous change. By rigorous I mean going right down to the ambiguous logical roots of the number system. Any professional mathematician who is not also an analytical bigot knows that the foundations of analysis are in a terrible mess. And any scientifically literate mathematician who follows what mathematical physicists are doing with this analysis and who criticizes them for their bold use of their dangerous mathematical machinery, proclaims himself a mathematical bigot of the first magnitude.

Would a summary of further advice from other great men of mathematics prove of service to the college teacher? If so, perhaps the National Council or the Mathematical Association of America would be willing to sponsor such a project separately or jointly. The American Society for Engineering Education published such a manual for its members in 1950.[16] Many teachers are acquainted with similar publications prepared by Massachusetts Institute of Technology[17] and the Bureau of Naval Personnel.[18]

Learning in mathematics

Every teacher knows that learning is not an absorption process of the sponge type where the mind soaks up the material in which it has been immersed. Nor is learning achieved by a "pouring in" or a "telling" process in which the teacher assumes the responsibility of being the chief "pourer" or "teller." Yet, just how does one explain this process we call learning? In finding some explanation especially as it applies to mathematics, it should be of interest to consider what other teachers of mathematics have thought and written on this subject.

Alfred North Whitehead, in his *Aims of Education*, which is now available in almost every bookstore and in many drugstores, states that there are three cycles involved in learning. These he calls the stage of romance, the stage of precision, and the stage of generalization.[19] Mary Everest Boole, wife of the famous mathematician George Everest Boole, considers the following as basic: homage, attention, observation, analysis, antithesis, synthesis, contemplation, effacement, repose, judgment or classification.[20] Jacques Hadamard stresses not only the three stages suggested by Helmholtz and Poincaré, which are called preparation, incubation, and illumination, but feels that a fourth and final stage proposed by Poincaré, called verification, is also required.[21]

Further discussion of this important question is found in the *Twenty-first Yearbook* of the National Council. Every beginning college teacher will benefit from a study of this yearbook and of related references.[22]

Textbooks and library books

Textbooks are used almost without exception in all of our early college courses

[15] E. T. Bell, *The Handmaiden of the Sciences* (New York: Reynal and Hitchcock, 1937), p. 62.

[16] Fred C. Morris, *Effective Teaching, A Manual for Engineering Instructors* (New York: McGraw-Hill Book Company, 1950).

[17] R. D. Evans, *You and Your Students* (4th printing; Cambridge, Mass.: Massachusetts Institute of Technology, 1954).

[18] *Instructor Training* (Washington: Bureau of Naval Personnel, 1955).

Manual for Navy Instructors (Washington: Bureau of Naval Personnel, 1949).

[19] Alfred North Whitehead, *The Aims of Education and Other Essays* (New York: Macmillan Co., 1929; Mentor Books, The New American Library, 1949), pp. 27–40. For a brief description of this volume see W. L. Schaaf, The Mathematics Teacher, XLVIII (May 1955), p. 348.

[20] Mary E. Boole, *Preparation of the Child for Science* (Oxford, England: The Clarendon Press, 1904).

[21] J. Hadamard, *The Psychology of Invention in the Mathematical Field* (Princeton: Princeton University Press, 1949; paperbound edition reissued by Dover Publications, 1955).

[22] "The Learning of Mathematics, Its Theory and Practice," *Twenty-first Yearbook*, The National Council of Teachers of Mathematics, Washington, 1953.

today. Hence much of the outcome of our college teaching is related to and influenced by their content and organization. Miller was among those who showed great concern regarding the influence that textbooks were having on our teaching and cautioned against the harm that is done by the use of poor texts. He said:[23]

> Millions of mathematical textbooks are now being used by students of our land and they constitute a very important factor in the diffusion of mathematical knowledge. We as teachers cannot afford to be indifferent in regard to the enormous influence exerted by these textbooks, especially since each of us is likely to be called on to help to decide at least minor questions relating thereto. . . . A poor textbook should be regarded as a menace and not as a piece of harmless literature. Not only is there danger that its author will use it with his own classes but if he happens to be widely known and influential there is great danger that it will replace better books in different institutions. The information freely provided by the publishers and their agents is often not unbiased, and the testimonials furnished by those who gave the book only a hasty examination are usually of less than face value. Even experienced teachers frequently find that they failed to judge correctly the merits and demerits of a particular textbook before they actually used it in class.

The fact that textbooks may retard mathematical progress is contained in a remark recorded by Seidlin when he quotes one of the professors at Yale University who commented on a widely used textbook: "It [the textbook] may yet go down into history as the *great* retarder in the progress of mathematics, as the *eliminator of thought processes* in what naturally is the most thought-provoking subject."[24] Professor Richard Courant of New York University also expressed this concern in blaming the trend away from the study of mathematics in part on "dull and empty commercial textbooks" when he wrote:[25] "The isolation of research scientists, the pitiful scarcity of inspiring teachers, the host of dull and empty commercial textbooks and the general educational trend away from intellectual discipline have contributed to the anti-mathematical fashion in education."

Individual teachers and colleges can do little to overcome the effects of the poor textbook on our teaching. However our mathematical organizations can and should give this matter serious study. Proper standards could be set up and good practices indicated. Authors and publishers would gladly cooperate. Miller echoed this proposal when he wrote in 1916:[26] "In these days of state, national, and even international conferences of mathematicians such big questions as the textbook problem may well come up for consideration even if it has thus far remained practically among the unregulated educational problems."

On the other side of our picture we have the matter of making greater use of the books on the mathematics shelves of our libraries. Is it true that our students rarely become acquainted with or use mathematics books in our college libraries during their first two years of study? If so, is this due to the fact that their own text is regarded as all-sufficient and no supplementary references are needed? Or is it true that there are few books published which do attract the attention and interest of our students? Or are our students so lacking in mathematical maturity that they are unable to comprehend some of the basic literature of mathematics? Are the students in the arts and other sciences usually limited to the study of a single text? Answers to these and related questions have an important bearing on how we can and should improve the teaching of college mathematics.

Instructional aids

Can instructional aids contribute to the improvement of our teaching? If so, where and how?

[23] G. A. Miller, "History and Use of Mathematical Textbooks," *School and Society*, IV (December 16, 1915), pp. 918–919.

[24] Seidlin, *op. cit.*, p. 78.

[25] Foreword to M. Kline, *Mathematics in Western Culture* (New York: Oxford University Press, 1953).

[26] *Op. cit.*, pp. 918–919.

Models of surfaces and polyhedrons always arouse greater interest than do the illustrations in the book. At the present time, however, commercial specimens are almost non-existent. There was a time when many good models and devices could be imported from abroad but it is doubtful that such commercial models will be available in the near future.

One solution would be to produce sets of slides in 3-D. Several different views of the same model could be made and in some cases color could be used to advantage.

We could also use some short motion pictures illustrating various loci and envelopes. Our books contain many references to "a point which moves" and while students may construct some of the simpler loci on graph paper, they frequently do not discover some of the properties of other interesting functions from the illustrations in the text. Would not a ten- or twenty-foot motion picture film, possibly prepared by the animation method, give a better impression of the path of the point? Another solution would be to prepare sets of five or ten 35 mm slides which would show the consecutive steps or stages in the construction of a curve. The effectiveness of such pictures can only be discovered after trying a few and comparing results.

An interesting introduction to curve tracing can be produced from a commercial toy known as the "hoot-nanny." This toy[27] and a book[28] describing the mathematics of these curves can be secured at a cost of no more than five dollars.

That a machine can be used for solving equations is a thought that is also new to many students of mathematics. Cundy and Rollett describe a simple model of a machine which will "solve simultaneous equations by using magnetically linked electric circuits," which they state "can be made without too much difficulty."[29] A crude machine for obtaining solutions of cubic equations called a hydrostatic equation-solver is also described by the same authors.

Some instructors of larger classes have experimented with the overhead projector. This makes it possible for one to face the class while material or diagrams on a celluloid surface are projected on an overhead screen. Successive stages for constructing certain curves can be produced on individual sheets of celluloid and overlaid on one another for effective projection purposes.

Stencil graph charts for preparing rectangular and polar coordinate grids on the blackboard have become pieces of valuable equipment in many classrooms. Charts showing logarithms and trigonometric functions to four places are also available commercially. Perhaps other well-prepared charts could render a valuable service in the teaching of certain topics.

Some pieces of demonstration equipment such as a Galton quincunx for demonstrating the normal curve and working models of certain linkages can serve an important function. A few pieces of wire and a little soap solution for demonstrating minimal surfaces can be worth more than a thousand words or a dozen pictures. The contribution that all such models and equipment can make to good teaching depends on how the teacher decides to use them.

TESTS

No one will deny that the kind and amount of testing we carry on in our classes can tell us a great deal concerning how much the student has learned about

[27] Available at many toy stores or from The Northern Signal Company, 120 N. Broadway, Milwaukee, Wisconsin.

[28] Robert B. Moritz, *Cyclic-Harmonic Curves, A Study in Polar Coordinates* (Seattle: University of Washington Publications, 1923).

[29] H. M. Cundy and A. P. Rollett, *Mathematical Models* (Oxford, England: The Clarendon Press, 1952), pp. 192–194.

mathematics. Such tests can also provide us with an indication of the effectiveness of our teaching.

Is an examination consisting of four or five problems the most effective way of testing the student's knowledge and progress? How are the problems chosen, and should they be scored as right or wrong, or should some credit be given for method, figure, accuracy, and other details? If so, how much?

In mathematics especially, could not more be learned about the actual amount of learning if we had several batteries of well-constructed tests prepared by a testing agency on a national scale? Multiple-choice tests can be scored easily by simple methods or by machine. "It is readily seen" that the frequency of certain incorrect answers will indicate areas in which further instruction is needed.

Suggestions on test construction and administration may be found in the previously mentioned publications of the Massachusetts Institute of Technology and the Bureau of Naval Personnel. It is hoped that some of the modern methods of testing will soon become common practice in our mathematics classes.

The professional background and training of college teachers of mathematics

"College teaching is the only major learned profession for which there does not exist a well-defined program of preparation directed toward developing the skills which it is essential for the practitioner to possess."[30] This statement appeared in the report of the President's Commission on Higher Education in 1947. According to Kelly, those who employ our Ph.D. graduates as college teachers are not satisfied with the training these graduates have received. Graduate schools, on the other hand, maintain that they are doing a satisfactory job of preparing prospective college teachers.

Kelly's report describes a study conducted by the University of Chicago's Committee on the Preparation of Teachers in 1948. Presidents of 850 universities, colleges, teachers colleges, and technical schools were asked to give their estimate of the strengths and weaknesses of college teachers as trained at that time in our graduate schools in America. Such strengths as the following were indicated:[31]

1. Well-prepared in his specialty.
2. Competent as a research scholar.
3. Generally high native intelligence.
4. Generally sincerely devoted to his scholarly interests.

Some of the weaknesses were also listed:[32]

1. Personal traits: colorless, queer, poor attitude toward teaching, doesn't like young people, the top-notchers are choosing other callings.
2. Too narrowly trained—too much specialization.
3. Interest is centered in research and not in teaching.
4. Lacks specific training for teaching. He has little knowledge of the learning process, of the place of motivation or the importance of self-direction. He thinks telling is teaching. He lacks effective techniques of presentation. He talks over the heads of his students. He lacks an understanding of higher education as an agency of democratic society.

Suggestions for improving the training of prospective college teachers were made by these employers as follows:[33]

1. Select a contingent of graduate students on the bases of their talents and aptitudes for college teaching.
2. Provide better for personal adjustments and personality development of prospective college teachers.
3. Adopt a graduate curriculum dictated by the needs of prospective college teachers.
4. Give teaching academic respectability comparable with research on the university campus.
5. Educate prospective college teachers in the science and art of teaching.

[30] Fred J. Kelly, *Toward Better College Teaching* (Washington: U. S. Government Printing Office, 1950), p. 1.

[31] *Ibid.*, p. 4.
[32] *Ibid.*, pp. 5–7.
[33] *Ibid.*, pp. 8–12.

The head of the mathematics department in one of our leading universities reflected the same feeling when he stated: "I think there should be more instruction in collegiate mathematical teaching required of prospective Ph.D.'s. Just because a person has a Ph.D. does not mean that he can teach. Small colleges make a mistake when they hire a weak Ph.D. in place of someone who may never become a Ph.D. but is a good and devoted teacher and has had considerable graduate mathematics beyond the elementary level."

Miller links the extensive use of the textbook in our mathematics classes to poor preparation of our teachers when he says:[34] "Our slavish dependence on the textbook is largely a result of poorly prepared teachers. The students should feel that they are studying subjects under the wise guidance of a teacher rather than the wise guidance of a textbook. . . . The remark of Max Simon to the effect that the logarithmic table is the only absolutely necessary mathematical handbook of the student should convey a useful lesson to many teachers."

In interviewing the candidate for a college position, could not the employer consider the candidate's familiarity with the cultural, historical, and pedagogical literature in mathematics as well as his interest in certain fields of pure research? Required reading for such prospective candidates could include books by Courant, Hadamard, Poincaré, Polya, Wertheimer, Bell, Kline, Schaaf and Synge.[35] A reading course based on these and similar books and some attention to problems of teaching mathematics should be required of doctoral candidates during the first or second year of graduate study. Remarkable improvement in undergraduate instruction could be the result in those universities in which these candidates also serve as assistant instructors during some of their years of graduate study.

Conclusion

We conclude this paper with the following quotation from F. D. Murnaghan of Johns Hopkins University:

> I know of no way in which we can serve our country better than by unswerving devotion to a crusade for the improvement of the teaching of college mathematics, and of no undertaking in which success could be more rewarding.

Who wants to join the crusade?

[34] G. A. Miller, "The Training of Mathematics Teachers," *School Science and Mathematics*, XV (January 1915), p. 11.

[35] Richard Courant and Herbert Robbins, *What Is Mathematics?* (London: Oxford University Press, 1941).

Jacques Hadamard, *The Psychology of Invention in the Mathematical Field* (Princeton: Princeton University Press, 1949; also New York: Dover Publications, Inc., 1955).

H. Poincaré, *Science and Method* (New York: Dover Publications, Inc., 1955).

G. Polya, *Mathematics and Plausible Reasoning* (Princeton: Princeton University Press, 1954), vols. I and II.

Max Wertheimer, *Productive Thinking* (New York: Harper and Brothers, 1945).

E. T. Bell, *Mathematics, Queen and Servant of Science* (New York: McGraw-Hill Book Co., 1951).

Morris Kline, *Mathematics in Modern Culture* (New York: Oxford University Press, 1953).

William L. Schaaf (ed.), *Mathematics, Our Great Heritage*, Essays on the Nature and Cultural Significance of Mathematics (New York: Harper and Brothers, 1948).

J. Synge, *Science: Sense and Nonsense* (New York: Norton, 1951).

"Fortunately, creative minds forget dogmatic philosophical beliefs whenever adherence to them would impede constructive achievement. For scholars and layman it is not philosophy but active experience in mathematics itself that alone can answer the question: What is mathematics?"—*taken from* What Is Mathematics? *by Richard Courant and Herbert Robbins, page xix.*

For a better mathematics program
2) In high-school geometry

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Euclidean geometry is now under attack from well-informed centers. "What should be done about it?" is a very real question for the high-school teacher.

THE QUESTION to be considered in this article is, "What can and/or should be done to improve the teaching of high-school geometry?"

The form used for phrasing the question might well provide us with that comfortable assurance which hunters must surely enjoy as, armed with both shotgun and rifle, they tread amidst the trees of the forest and the grain of the field in search of game. As we thread our way through the consequences of experience and the implications of experimentation there is indeed comfort in the thought that although we may have difficulty in bringing down significant game when we fire our professional rifle, "What Can Be Done?", surely we can achieve some type of success when we fire with our shotgun, "What Should Be Done?" Indeed, seldom does one miss with this trusty gun, since it adapts so easily to individual aiming techniques. Firing wildly with both guns, whether consecutive discharge or simultaneous volley, we might even be able to scatter pellets of inquisitiveness that would bring to earth at least a few desirable morsels of thought worthy of careful consideration and mature deliberation. We should beware, however, lest the quail of our quarry might also have cause to strut his scorn, the goose to honk his derision, and the squirrel to flick his contempt at such amateurish efforts when the potential returns of professional finesse and skill are so full-flavored and wholesome.

In our professional safari let us, with some degree of temerity, test our educational marksmanship by deliberately choosing to fire our guns one at a time and in consecutive order: first the shotgun, "What Should Be Done?"; and second the rifle, "What Can Be Done?" Furthermore, let us load each gun with a double charge of inquisitiveness. Thus, as we contemplate our equipment, we find our professional guns purposefully loaded as follows and named in deliberate choice of order for firing:

The Shotgun

1. What should be done to improve the curriculum in high-school geometry?
2. What should be done to improve the techniques of teaching high-school geometry?

The Rifle

1. What can be done by us *as individual teachers* to improve the teaching of high-school geometry?
2. What can be done by us *as an organized group* to improve the teaching of high-school geometry?

Now that we are ready "with stately mien" to stalk our professional game, where to hunt becomes the paramount question. What areas offer the greatest opportunities for successful hunting? There cannot be much doubt that the area of greatest promise for satisfying and significant results from the use of our professional rifle must be the future program of geometry in our high schools. Furthermore, the prospect of the future is, at all times, most intelligently contemplated through the perspective of the past and the substance of the present. Thus it would seem that the most promising areas for hunting with our professional shotgun are those of past experience and present practice in high-school geometry. Let us go forth, then, hoping that we may have some significant success in our professional hunting expedition.

What should be done to improve the curriculum in high-school geometry?

Why should geometry occupy an important place in the high school curriculum? Traditionally, geometry has for many centuries occupied a significant place in the instructional programs of the institutions of our culture. The expanding philosophy of education has long since taught us that tradition, as a sole criterion, should not carry a great deal of weight in the selection of instructional materials designed to meet the demands of a modern educational philosophy. This fact, recognized early by curriculum designers and committee reporters, was finally substantially supported in the field of mathematics education by the significant organization of the aims of mathematical instruction into the three categories of *practical, disciplinary,* and *cultural,* as proposed in the 1923 Report of the National Committee on Mathematical Requirements. Herein was believed to lie the framework of a guiding philosophy in the selection of curriculum content in mathematics, which was designed to effect the most significant synthesis of the counsel from the tradition of the past, the challenge from the experience of the present, and the expectation from the anticipation of the future. The endorsement and extended interpretation, by influential writers and committee reports, of these aims have continued through the three decades since the publication of the report. Insofar as geometry can contribute to these aims of instruction in mathematics, it has a place in the curriculum of the high school.

Among the accepted aims of instruction in geometry we can list: (1) Familiarity with the geometric forms common to our environment and the basic properties and relationships of these forms; (2) Appreciation of the art characteristics of these forms; (3) Understanding the basic principles of direct and indirect measurement; (4) Development of space perception and spatial imagination. Why should the students in our high schools be denied the opportunities which authoritative thinking says they should have? We live in a space of three dimensions, not two dimensions. Why should we expect these students to become familiar with the geometric forms common to their environment and with the basic properties and relationships of these forms when they are not given an opportunity to study them under informed supervision? Why should we expect them to get any basic information concerning the art characteristics of these forms? About the only opportunity afforded the vast majority of our high-school pupils to learn anything about the three-dimensional space in which they live is through simple intuition and commonplace measurement. Long years ago geometry freed itself from the restricting shackles of mere earth measurement. Intelligent understanding and significant appreciation call for more than the superficiality of measurement and probability of intuition. It is difficult to see how adequate intelligent training in the essentials of space perception and spatial imagination can be accomplished through a

confused fusion or a tantalizing tandem arrangement of selected excerpts from the traditional programs in plane and solid geometry. Is it possible that we should recognize that if we really wish to develop space perception and spatial imagination in an effective manner, we may need to teach a different type of geometry from that of the traditional Euclidean pattern? Or, at least, maybe we should refuse to follow the dictates of that traditional pattern of instruction which, for some mysterious reason, has decreed that the academic geometrical experiences of the high-school pupil should be primarily two dimensional while the natural geometrical experiences of this same pupil are primarily three dimensional.

As early as 1894 the Committee of Ten recommended that "instruction in concrete geometry, with numerous exercises, be introduced into the grammar school . . . to familiarize the pupil with the facts of plane and solid geometry." The reports of all significant committees since the time of that early report have supported and strengthened this recommendation. The result has been a somewhat heterogeneous distribution of geometric materials throughout the subject content of the mathematics of the elementary school and the junior high school. The complete picture of heterogeneity of treatment is not fully comprehended until we recognize the fact that this informal treatment of selected geometric concepts and techniques has made its contribution to the "correlation of confusion" in the junior high school.

What should be the basic philosophy for the selection and organization of geometric concepts, principles, and techniques for a truly significant program in intuitive geometry? The National Committee on Mathematical Requirements in its 1923 report said: "The work in intuitive geometry should make the pupils familiar with the elementary ideas concerning geometric forms in the plane and in space with respect to shape, size, and position, i.e., this informal work should be so organized as to make it a gradual approach to, and provide a foundation for, the subsequent work in demonstrative geometry." Before this statement is acceptable as a guiding criterion, it would seem desirable to supplement it by the further stipulation that this informal work in geometry should also provide a means for geometry to make its peculiar contribution to that functional competence in mathematics which the Commission on Post-War Plans stated should be guaranteed to all educable individuals. Oddly enough, although this appended stipulation was not a part of the original statement, it has seemed to be the principal criterion through the years in the selection, organization, and presentation of geometric materials in the elementary school and junior high school. An unfortunate outcome of such a pattern of thinking, however, has been that the treatment of geometry in the junior high school has seemed to become imbedded in a context of terminal-course superficiality. In the informal program of intuitive geometry, what should constitute the proper balance of emphasis between the criterion of provision for functional competence and the criterion of provision for a foundation for subsequent work in geometry? Should the informal geometry of the earlier school years constitute a significant background for the more formal geometry of the high-school years in the sense that it would become a required, or even a desired, prerequisite? Should some of the subject matter of geometry be removed from the area of formal treatment and be treated only informally? Should we attempt to organize syllabi in both formal and informal geometry which would recognize and delete undesirable repetitious treatment of concepts, principles, and techniques? Should careful thought be given to grade placement of geometric materials from the elementary grades to the junior college? Should we give careful study to the proposal that maybe there

is a place for some type of formal study of demonstrative geometry in the earlier grades? This thought was first given expression as early as 1899 when the Committee on College Entrance Requirements recommended an "Introduction to Demonstrative Geometry" for the eighth grade. Possibly this is an idea which demands exploration by careful experimentation. Should we recognize that the effective teaching of geometry probably is not merely a problem of the tenth grade alone, but a problem which casts its shadow of proper selection, grade placement, and treatment of geometrical content over the entire span of educational experience from the elementary school to the junior college? While we shall continue to emphasize the deductive nature of geometry, should we not make a more intelligently planned effort to capitalize on the opportunities afforded by experimental intuitive geometry to shape the foundation upon which might be built a framework of thinking that would be characterized by full realization of the powers of inductive inference, as well as by a clear comprehension of its inherent dangers? Whether we openly recognize it or not, constructive intuition will always be an intrinsically vital element of geometrical thought.

From the days of the earliest committee reports on the teaching of geometry, there has always been a rather clear-cut dichotomy in the grouping of the aims and objectives of geometrical instruction, namely: (1) Those which contribute to the development of logical thinking; and (2) Those which formulate a structure of significant information concerning geometric facts. The emergence of a pragmatic philosophy of education has resulted in the recognition of a third important group: (3) Those which emphasize the application of geometrical principles and techniques to the better interpretation of our environment.

In 1935 the "Third Report of the Committee on Geometry" stated: "Teachers agree that the main outcomes of demonstrative geometry pertain to logical thinking."[1] The essentially contemporaneous studies by Fawcett (1938) on "The Nature of Proof," Ulmer (1939) on "Teaching Geometry to Cultivate Reflective Thinking," and Gadske (1940) on "Demonstrative Geometry as a Means for Improving Critical Thinking" became spearheads of a movement to place a great deal of emphasis on the use of geometry as a vehicle for "the development of logical thinking" and, supposedly, for more meaningful instruction and greater transfer value, as well as to supplement regular geometric materials with non-geometric materials from various types of life situations. This movement has exercised a tremendous influence on the geometry program of the high school. An authoritative writer in a recent article states: "It is heartening to observe that now . . . all widely used geometry textbooks give some attention to reasoning in non-geometric situations. . . . Here is evidence of an emerging practice in the teaching of geometry."[2] This writer goes on to list eleven items which, to him, appear to "indicate points of current emphasis in teaching geometry."[3] Only three of these eleven items in any way seem to take cognizance of the fact that there might be a body of geometrical knowledge which is of considerable significance at the high-school level of instruction. Should we not become cautious lest, in shaping the structure of high-school geometry, we might be beginning to allow the tail to wag the dog? Should we not pause long enough in our planning of the geometry program for re-examination of the purposes for which geometry should find a place in the high-school curriculum?

There are those who assert in very positive terms that algebra, properly taught,

[1] Ralph Beatley, "Third Report of the Committee on Geometry," THE MATHEMATICS TEACHER, XXVIII (March 1935), p. 334.

[2] Le Roy H. Schnell, "The New Emphasis in Teaching Geometry," *Emerging Practices in Mathematics Education, Twenty-second Yearbook* (Washington: The National Council of Teachers of Mathematics, 1954), p. 274.

[3] *Ibid.*, pp. 275–276.

not only "can help to develop in students many of the same concepts of logic as geometry," but, actually, that algebra can "provide easier examples of precise mathematical proof than does geometry." In fact, more and more frequently we are finding informed opinion in the field of mathematics insisting that the teaching of mathematics should give more explicit recognition to the fact that the use of deductive logic is a significant characteristic of all mathematics and does not typify geometry alone. What should be the relative emphasis on the geometric and non-geometric content of the geometry curriculum? What should be the non-reducible minimum of geometric content? What emphasis should be given to the nature and use of converses, inverses, and contrapositives? What use should be made of the indirect type of proof? What emphasis should be given to the analytic approach to geometric proof? Should we study very carefully the new emphases concerning the content of high-school algebra for their implications as to possible modifications of the geometry program? Should the use of such non-mathematical materials as analysis of propaganda, advertisements, speeches, etc., be any more prominent in the geometry class than they are in any other class in mathematics? Should they be given any more than minor emphasis in any mathematics class, and that merely for purposes of illustration and suggestive application of principles? Why should the geometry program or, indeed, the mathematics program attempt to usurp from other areas of human endeavor the responsibility for teaching applications of the techniques of deductive thinking any more than it would attempt to usurp from science and engineering the responsibility for teaching applications of all significant mathematical techniques? Certainly geometry has a responsibility to contribute toward a clearer comprehension of the intricacies of deductive thinking. That, however, is not its sole responsibility and possibly not its major responsibility as a significant part of the high-school curriculum. No other subject in the curriculum shares with it the responsibilities for building a structure of significant geometric knowledge and for giving intelligent guidance in the application of geometrical principles and techniques. Should we not attempt to clarify aims and objectives for the teaching of geometry?

Flitting about on the horizon of present-day thinking concerning the high-school program in mathematics there is another pertinent problem which, though still somewhat amorphous, is already demanding enlightened consideration. There is a rather strong conviction, held by a good many individuals, that the newer developments in mathematics have many implications for modification of the content of the mathematics program in the high school and in the junior college. Concrete evidence supporting this statement may be found in the recommendations made at the Symposium on Teacher Education in Mathematics held at the University of Wisconsin in 1952, the current program of summer institutes for teachers of mathematics sponsored by the National Science Foundation, and the forthcoming yearbook on *Insights into Modern Mathematics* sponsored by The National Council of Teachers of Mathematics.

Should we not also recognize that there is more to the teaching of mathematics, an alive and rapidly growing area of intellectual endeavor, than the mere stirring of the ashes and bones of tradition? With the great strides in geometrical thinking that have taken place during the twentieth century, should we not consider it a bit paradoxical that much of our thinking concerning the subject-matter content of high-school geometry antedates the Christian era? In this modern age projection, inversion, stretching, bending, and compression, as well as translation and rotation, are significant in the

consideration of geometrical forms and configurations. In fact, we frequently hear projective geometry referred to as "pure geometry," and topology as "the most general of all geometries." Their basic concern is with the fundamental subject matter of geometry, namely, the spatial relations between points, lines, and planes, free from any contaminating restrictions of number and measurement. Should the geometry program of the high school provide basic opportunities for becoming familiar with fundamental geometric properties, principles, and techniques other than those that are primarily metric in nature? For the maximum contribution the study of geometry might make toward a clearer comprehension of the postulational-deductive method, should we not thoughtfully consider the use of non-Euclidean materials with at least as much emphasis as non-geometric materials? Someone has said: "Yes, the modern world was created on a certain day, June 5, 1637!" If the Cartesian system of thinking about geometrical properties has such great potential significance, why should the high-school pupil be deprived of the opportunity of becoming familiar with some of its basic techniques? Why should we not attempt to extend the horizons of elementary geometry by garnering from the various areas of geometrical thinking at least those elements which adapt themselves to non-specialist comprehension?

The tempo of development along the avenues of national and international travel has become terrific. When we use obsolescent modes of conveyance we endanger not only our own lives but also the lives of others who seek both pleasure and profit from this travel. Just so, the tempo of mathematical growth along the avenues of intellectual thought has become terrific. When we use obsolescent materials and techniques we jeopardize not only our own mental growth, but also the mental growth of others who seek both information and inspiration from their study of mathematics.

What should be done to improve the techniques of teaching high-school geometry?

Not all problems related to the improvement of instruction in high-school geometry are confined to the area of curriculum content. Many of them are concerned with the techniques of teaching. While it is quite true that we can waste time very effectively by skillfully teaching unimportant things, it is probably more distressingly true that, most assuredly, we can destroy interest by incompetent, stereotyped interpretations of even the most significant materials.

During the highly impressionistic days of the high-school period of education, the shaping of an effective program of developmental teaching for any new subject-matter area is an instructional problem of serious importance. There is most likely no single best developmental pattern for guiding a group of high-school pupils in the formal study of geometry. This fact emphasizes the necessity for attempting to determine those criteria that should be considered by the teacher in the selection and shaping of the most effective treatment of geometry for a particular group of pupils. What provisions should be made for the recognition of individual differences among pupils? What background of information should be assumed as the foundation upon which to build the formal program in geometry? What should constitute the proper balance between insistence upon rigor and acquiescence to intuition in the shaping of geometric proof? What should be the desirable and acceptable characteristics of proof intelligently adjusted to the limitations of mental maturity and intellectual insight of the particular age-group for which the program in high-school geometry is designed? Other than consistency and completeness, what should

control the formulation of the postulates for a truly significant program in high-school geometry? What recourse to non-Euclidean geometries should be had as an aid in emphasizing the dependence of conclusions upon hypotheses? How much emphasis should be given to the significant distinction between validity and truth in the consideration of geometric theorems? How much attention should be given to the significance of undefined elements and definition in the structure of geometry? To what extent should the application of algebraic techniques be encouraged in the solving of geometric problems? How far should we go in the effort to encourage and direct pupils to formulate their own proofs of theorems? What emphasis should be placed on memorization or non-memorization in the study of geometry? What emphasis should be placed on exercises in the construction of geometric configurations and other forms of original exercises? How much attention should be given to discovery by analysis and the discussions of limitations in all types of original exercises, particularly construction exercises? What should constitute a sane use of non-geometric materials as an aid to securing more fruitful motivation and more effective transfer? In the teaching of geometry, what emphasis should be given to drill? What are the basic understandings and fundamental techniques that should be developed in geometry? What should constitute a sensible program in the use of multisensory aids in the teaching of geometry? To what extent should the program in geometry become a laboratory course?

While the conversion of a mathematics classroom into a laboratory can be very expensive, both in money and time, there are arguments in favor of the use of teacher-made and student-made devices as aids in teaching. It has been suggested that such devices group themselves rather descriptively into four major categories: demonstration aids, developmental aids, analysis aids, and projects. While the laboratory approach to the study of geometry adapts itself very effectively to individual differences among pupils and has distinct potentialities in the areas of motivation and transfer, it seems rather heavily weighted with the limitations of financial cost, manual dexterity, and the tedium of unavoidable detail.

In spite of the fact that theories concerning transfer of training take shape only in the context of changing theories of learning, it is generally accepted as true that transfer does take place, is not automatic, and its amount and effectiveness are dependent upon many different factors. What techniques of teaching should be employed to produce the most effective transfer of significant learning from the study of geometry? For maximum transfer what relative emphasis should be given to reconstruction and generalization of experience, exploration, discovery, and drill? What basic geometric concepts and principles have the greatest potential for significant transfer to other areas? What should be the relative emphasis on understanding and technical proficiency for the most effective transfer?

Among the important responsibilities that a teacher of geometry has to assume is setting up a significant program of evaluation. Such a program is no longer couched in a pattern of mere ability to recall information and to perform certain skills. Other desirable outcomes of instruction which need to be measured are understanding, attitude, appreciation, and capacity for use of acquired information and skills. Providing behavior situations designed to procure valid measures of such significant instruction objectives is a teaching technique of major importance. Furthermore, the functional use of evaluative procedures in intelligent guidance takes on major significance in an elective subject area such as geometry. There is much more to evaluation than the mere measurement of achievement. What criteria should the teacher of high-school geometry use as an aid in the construc-

tion of an efficient evaluation program? What should be the relative weights given to teacher judgments, teacher-made tests, and commercially-produced tests? What should constitute satisfactory bases for choosing between essay-type tests and short-answer-type tests? As a basis for the formation of teacher judgments, what should be the relative importance given to oral class recitations, comparative class observations, outside class assignments, written class assignments, personal interviews, anecdotal records, and case studies?

Such questions as those raised in the preceding paragraphs can but remind us that we must be alert at all times to the many problems that arise in our classrooms. Not only must we keep in contact with new developments in the subject matter of the area in which we teach, but we must also be awake to new developments in the psychology of learning and to significant techniques of teaching that evolve from experimentation and experience. We must keep ourselves reminded constantly that, whatever we teach, our worst enemies are most likely carelessness, inaccuracy, forgetfulness, slovenliness, and ignorance.

This should be done to improve the teaching of high-school geometry

By the firing of the two charges of "curriculum" and "techniques of teaching" from our professional shotgun "What Should Be Done?" we have projected many pellets of inquisitiveness amidst coveys of curriculum proposals and flights of learning philosophies. Let us now pause long enough to see if we can discover what our faithful retrievers, implication and interpretation, might help us to gather into our gamebag of professional resolution. As teachers of geometry we should resolve so to fashion our instructional program that geometry will receive the recognition it deserves as an essential element of the school curriculum. This means that we should, at all times, strive to maintain a proper balance in the relative emphasis on the practical, cultural, and disciplinary values to be derived from the study of geometry. We should constantly seek to keep ourselves informed not only concerning those teaching techniques which the latest developments in psychology recommend as the most effective, but also those new developments in the subject matter of geometry which competent mathematicians are able to interpret for intelligent comprehension by those who, while not specialists, are basically informed in geometry. Geometry is a live subject, constantly growing in the challenge of modern development; we should make every effort to teach it as such. Furthermore, the study of geometry should offer rich opportunities to every pupil for interpretation of the space in which he lives, and also for basic comprehension of the significance of postulates and the validity of conclusions. We should revise our program in geometry to include basic experiences and information from three-space and from non-Euclidean geometries. This does not mean the mere extension by analogy to, or excerpt from, solid geometry, but a systematic study of the space of our environment, whether it be one-space two-space, three-space, or even four-space, using time as one of the basic dimensions. Nor, does it mean a philosophical treatment of the properties of non-Euclidean geometries, but rather the simple drawing of a few contrasts in the basic postulates and calling attention to where one might expect to find both agreements and disagreements between these contrasting yet self-consistent geometries. We should not allow ourselves to become moss-covered with tradition, but should strive at all times to seek, both among the old and the new, those basic understandings and appreciations in geometry the development of which is just as significant a teaching responsibility as is that of developing certain selected concepts, principles, and skills. Furthermore, intensive study should be given to the

problem of valid evaluation of such instruction.

Any syllabus, or textbook series, designed for use in the elementary grades or for the junior high school will have, at least from the third grade on up, a distribution of geometric content. Is the selection and placement of this material in accordance with a carefully designed plan or is it largely whimsical in nature? We should devote some careful study to the problems of grade placement of geometric materials, the relative distribution and interrelationship between materials suitable for informal treatment and those suitable only for formal treatment, the relative importance of induction and deduction in argumentation, and the relative values of intuition and formal proof. We should give very careful thought to the determination of the non-reducible minimum of geometric subject matter for which the school program in geometry should be held responsible. This means that we should keep in mind at all times that, while there are many instructional responsibilities which geometry should share with other mathematical units of the curriculum, the development of geometrical concepts, skills, understandings, and appreciations is the sole responsibility of the geometry program. We should therefore re-examine and re-state the purposes for which geometry should find a place in the high-school curriculum. We should attempt to clarify the aims and objectives for the teaching of geometry not only in the high school, but in the elementary grades and junior high school as well.

The teacher of geometry not only needs to be well-informed in the subject matter of geometry and in the psychology of learning, but he should be convinced of the rich potentiality of geometry as a curriculum element. His pattern of teaching should, at all times, be inspirational rather than mechanical. He should strive constantly to help his pupils experience the thrill of discovery and the challenge of validation, to make them conscious of the probability of induction and the security of deduction, and to lead them to appreciate the satisfaction of organized information and the confidence of independent thought.

What can be done by us as individual teachers to improve the teaching of high-school geometry?

Up to this point in our professional safari we have enjoyed the rather accommodating direction of such genial guides as curiosity, concern, and conviction. They are always willing and able to serve as guides when we seek those things that should be done. They are not always too particular in their prescriptions as to what kind or how much game one might bag. In fact, they can be quite ruthless and even unsportsmanlike in their behavior. It is also true that so far our marksmanship has not had to be too sharp, and it has not been too difficult to bag abundant game.

From here on, our hunting will be quite different. In the first place, the rifle "What Can Be Done?" is of a distinctly different bore from that of the shotgun, "What Should Be Done?" In the second place, the game presents much greater challenge in the quest but far more reward in the kill. It is more individualistic, much more elusive, and far more difficult to bag. Our aim must, of necessity, be more discriminating and our marksmanship more accurate. Furthermore, if we hope to have any significant success on this hunting jaunt, we shall have to be adventurous enough to seek the bold direction and cool counsel of such stalwart guides as daring, determination, and desire.

A question of major concern to us should be: "What can be done by us *as individual teachers* to improve the teaching of high-school geometry?" Supported and encouraged by determination and desire we can all fashion programs of individual study designed to keep ourselves informed concerning new developments in the sub-

ject matter of geometry. We can constantly seek ways and means of transmitting significant aspects of this new knowledge to the uninitiated and uninformed. We can keep abreast of psychological study and experimentation so that we might profit from new ideas and tested techniques of motivation, interpretation, and evaluation. We can resolve not to allow ourselves to become stagnant in the comfortable self-complacency of conformity and acquiescence, but to keep alert to the challenge of change and progress. We can be persistent in our search for more effective methods of presenting geometry as a living and growing subject and not as a subject whose most significant content is to be gleaned from the archives of the past. We can give careful thought to the practical, disciplinary, and cultural values to be derived from the study of geometry and strive so to organize our instructional programs that we shall not only give due recognition to each group of values but also maintain a sane balance between the three. We can engage in experimentation on such problems as those of grade placement of geometric materials, motivation procedures, curriculum organization, evaluation patterns, and teaching techniques. Many of these problems are too big to be tackled with any degree of finality by individuals, but such individual efforts can have great potential as pilot studies for the much needed more elaborate investigations. We can direct our students so that they might have the opportunity to derive from their study of geometry those characteristics of postulational-deductive thinking which such study should provide them. At the same time we can recognize that we have a definite responsibility to keep in mind that first and foremost the program in geometry is, indeed, a program in geometry and not a substitute course in logic. We can so shape our instructional program that there will be planned effort to help our pupils, through generalization and rationalization, get some appreciation of geometry as a significant element in the closely knit, but highly variegated, area of human knowledge known as mathematics. We can promote informational programs in our schools and communities to acquaint pupils and other members of the lay public with the significance of geometry and other mathematical subjects and to warn against the hidden dangers lurking behind the cloak of terminal study. We can keep actively cognizant of the problems involved in teaching geometry so that our students will derive maximum benefit from its study; so that they will enjoy the challenge of discovery and the confidence of proof; so that they will realize the possibilities, limitations, advantages, and disadvantages of induction and deduction; so that they will experience the subjective freedom of informality and the objective substance of formality. This means that we can and should strive to plan our program of instruction not only so that each pupil will be able to make the maximum transfer to his complete environment from his learning experiences in geometry, but also so he will be able to realize the comfortable security of a significant reservoir of geometrical knowledge.

What can be done by us as an organized group to improve the teaching of high-school geometry?

Archaeography is "a treatise or writing on antiquity or antiques." The program of study proposed for high-school geometry should never be allowed, either literally or figuratively, to approach synonymity with archaeography. For too many years we have been dangerously near allowing this to happen. It is a remarkable testimony to the inherent qualities of geometry as a significant instructional medium that its structure has so long and so well withstood the waves of criticism which have been directed against it. Of how much more educational moment might the high-school program in geometry become if the truly essential matter from the

past were inwrought with the richly significant matter of the present! There is a desperate need for a study, made under sponsorship of an organized authoritative group, to determine the most appropriate and challenging synthesis of these two highly important areas of geometric knowledge. Such a study can and should prescribe a pattern for the reorganization of the geometry program in our schools. No individual writer or experimenter can assume such authority or presume such influence as that essential for a truly effective study of this magnitude.

Possibly one of the most difficult to explain of the many paradoxes which bespatter intelligent curriculum construction is one which is distinctly peculiar to the geometry program. For many years now, accepted philosophy of curriculum construction has recognized among the significant aims of education the development of space perception, spatial imagination, and intelligent comprehension of the properties of and relationships between the geometric forms which characterize our environment. Geometry is the one subject in the curriculum which can and should provide academic spatial experiences commensurate with the demands for intelligent comprehension of our natural spatial experiences. It is very difficult to rationalize the relatively small emphasis our present courses of study in geometry give to the consideration of the basic concepts, fundamental principles, and more significant relationships of three-space—the space in which the vast majority of our daily actions, meditations, and interpretations take place. What is the most pertinent body of geometrical subject matter and what is its most effective form of organization for comprehension and interpretation of our natural spatial experiences? This is another question of major importance which needs to be considered in an atmosphere cleared of any prescriptions from tradition and charged only with the demands for the most significant contribution which the study of geometry can make to the education of the youth of our land.

Professional thinking has given some recognition to the educational values it has attached to geometry by distributing through the grades of the elementary school and the junior high school many geometric concepts, principles, and skills. Through informal and experimental experiences in these grades opportunity is provided for building a rather substantial background of geometrical knowledge before the pupil enters the more formal program of the high school. Are the materials used in our present program in geometry the most suitable for such treatment, and have they been allocated to the most appropriate grades? Is there a basis for sequence of pattern in the geometry program which leads from experiment to proof, from intuition to argumentation, from the informal to the formal? Grade placement and sequential arrangement of geometric materials, while difficult problems to contemplate, can be solved only through the deliberate and decisive action of an organized, authoritative group of interested workers devoted to the development of a program in geometry of maximum educational significance.

No matter how elaborate or constructive a plan might be drawn for improving the instruction in geometry in our schools, the fact still remains that the potential significance of geometry as an instructional medium lies in the hands of the teachers. No one can teach geometry unless he is well-informed in the subject matter of geometry and has an appreciation for its value as an element of the school curriculum. Furthermore, a well-informed teacher can do a more effective job of teaching if he is well versed in the psychology of learning, the techniques of presentation, the art of motivation, and the patterns of evaluation.

Authoritative group action should be taken in the specification of minimum standards of significant knowledge of subject matter and effective techniques of

instruction not only for teachers of geometry, but also for teachers in other areas of mathematics as well. One of the results of the study and action of this group should be the correction of the ridiculously low certification standards for teachers of mathematics that exist in some of our states. This same group should also take definite and positive action to procure a more informed program of guidance in our schools. High-school pupils have probably received, under the guise of professional guidance, more wrong information concerning the value of algebra and geometry than they have received about any other subjects in the school curriculum.

This discussion of some of the problems related to the improvement of instruction in geometry has, in no sense, attempted to be exhaustive. Rather its intent is merely suggestive. Some of the problems recognized lend themselves to individual consideration; many cannot be solved by disconnected individual effort, but demand the persistent consideration of authoritative group organization. The needed action, however, is not that of a group commissioned merely to survey, select, and recommend. Rather, what is needed is action by a group empowered to act with authority, delegated to represent the best thinking in the interest of significant instruction in geometry, and commissioned to study, ascertain, and prescribe. Whether individual or group, the action must be with serious and devoted intent, and the design must be to attack our problems with boldness and to pursue the resolution of these problems with persistent determination and driving desire.

Letter to the editor

Dear Sir,

I was interested in the comments and suggestions in the October 1955 issue of the TEACHER relative to book reviews. I have tried to do some thinking along these lines. I tend to agree, in part, and disagree elsewhere. I have the following suggestions:

1. No unsolicited review should be published.
2. When an individual has been requested to review a book, he is assumed to be competent; his review as submitted must not be changed without his consent.
3. It seems a needless waste of time to referee all reviews before publication. Where, in the opinion of the editors, a second opinion is advisable, two options are possible:
   a. The review is published, followed by comments by a referee, these being restricted to material appearing in the review.
   b. A second review is published simultaneously. It may be of value for the second reviewer to have seen the first review; there is also merit in having it written independently.
4. After publication of a review, the author (or possibly any reader) should have the opportunity to present in print refutation of statements charging factual error. There should be no opportunity allowed to discuss advantages or disadvantages of methods of treatment, order of topics, or other items which are a matter of opinion rather than of fact. Prior to printing such refutation, the reviewer should be consulted, and all errors should be included in the published material. If there is disagreement, in fairness to the readers, the opinion of one or more authorities should be given.

If these suggestions were followed, I think the review section would be of very great value. With a few exceptions, however, the past record of the teaching profession indicates an unwillingness to severely criticize a colleague.

I suggest the following as an interesting exercise for a class in secondary-school mathematics: Examine the reviews of high-school geometry (or algebra) texts printed anywhere during the past ten years, and state how many "bad" books were published. Compare the results with the opinions of perhaps fifty to one-hundred teachers.

Regards,
C. B. READ
University of Wichita
Wichita, Kansas

For a better mathematics program
3) In the junior high school

R. E. PINGRY, *University of Illinois, Urbana, Illinois.*
The improvement of content is not the only problem confronting those interested in the junior high school. Psychological problems, teacher shortages, and poor working conditions must be considered.

BEFORE BEGINNING my discussion of some problems of mathematics teaching in the junior high school, I should state what I mean by a junior high school. There are various ways of defining a junior high school. There are many different types of schools called junior high schools. Sometimes grades seven and eight are located in the high school building and called the junior high school. Sometimes grades seven and eight are located in the elementary school and called the junior high school. At other times separate school buildings and organizations are provided for grades seven, eight, and nine, or grades seven and eight.

At times the junior high school advocates add to their definitions certain requirements concerning the philosophy of education and objectives of teaching. Thus, a separate school building housing only grades seven, eight, and nine may or may not be a junior high school depending on the guiding educational philosophy of the school.

For this discussion of junior high school mathematics, however, I mean simply the mathematics usually taught or desired to be taught in grades seven, eight, and nine, no matter how the school is organized or what the school's guiding philosophy may be.

INCREASING JUNIOR HIGH SCHOOL POPULATION

In recent years the number of junior high schools, schools organized for grades seven, eight, and nine, has been increasing rapidly. As larger and larger numbers of students press into the secondary schools, school boards have found that junior high school building programs have frequently been necessary.

The New York State Department of Education reports that in grades seven to nine there were about 400,000 pupils enrolled in 1952–53. In 1956–57 the prediction obtained by use of actual birth statistics is that 600,000 pupils will be enrolled in the junior high school gradès. In the year 1965–66 the prediction is for 800,000 pupils or just double that for the year 1952–53.

The junior high school years are certainly critical in the educational experience of a student. Attitudes toward school and life are frequently fixed during these years. School experience during this period should be the best possible experience.

The problem of increasing enrollments combined with the problem of supply of qualified teachers and facilities for the junior high school is certainly one of the most significant and urgent problems in education today.

Individual differences

One of the chief instruction problems related to the junior high school is that of the great range of abilities, interests, physical development, and motivation of the students.

I have taught in junior high schools, and in my present work I visit regularly in junior high schools. One of the most striking observations I make as I visit these schools is that there is great variability in the many aspects of student development.

Frequently, as I stand in the corridor and watch the students pass between classes, I notice a strapping young man who is nearly six feet tall followed by a freckled faced lad who still looks like a sixth grader and speaks with the soprano voice of a choir boy. A young lady who might pass as a freshman coed in college, sits in the same room with a little girl in pigtails and ribbons.

These noticeable physical differences are typical of great differences in many other aspects. Many physical and mental characteristics have a very great range at about age 12. At the eighth grade level, students of 13 years of age may easily range in mental ability from a mental age of 7 to a mental age of 19. In one Illinois town the students in grade seven were measured in arithmetic ability, and they ranged from a grade level of 4.7 to 8.7. This was not in a school where there had been automatic promotion either, but in a school where there had been a rather high failure rate. In another school, students in grade eight ranged from grade levels 4 to 13 in mathematics achievement, a span of nine years. The chronological age span was only over four years with 97% of the class within one year of the median age.

Great differences also exist in the students' values, interests, and motivation. I cannot describe these differences in terms of a range of years, but possibly I can in terms of behavior.

Let me tell you about one eighth-grade class I taught. In the back row sat Tom. Now Tom was absent a lot, but a few days a month the truant officer managed to keep him in school. Tom was almost 16. He would soon be able to quit school. "Oh, glorious day," he thought. Tom's father didn't mind. He told Tom school wasn't any good. He couldn't see where it ever did him any good.

In the same class with Tom was Jack. Jack's mother and father introduced themselves early in the year. They wanted so much for Jack to do well in mathematics, because they knew how important it was, no matter what Jack's future work would be. Jack wanted to study, too. He liked his mathematics and came around after school to ask interesting questions that showed a lot of thought.

These great differences in native ability, level of achievement, and motivational level impose tremendous problems upon the junior high school mathematics teacher who meets four, five, or more classes a day with 30, 35, or, yes, even 40 students per class.

I know of no problem in teaching today of as much general concern to sincere teachers, especially junior high school teachers, as this problem of individual differences.

What does this problem mean to mathematics teaching?

Learning mathematics

Before we answer this question, let's take a look at the way students learn mathematics. Mathematics, as a discipline, is a complex structure of ideas. If a student is to learn these ideas he has to work up through this structure. He cannot enter this structure at any place he chooses. He has to come in at the ground floor and move up from one idea to the next.

It may be possible in history class for a student to learn a lot about the history of the American Indian without knowing the history of ancient China. He can study some chemistry without first having had physics. It is impossible, however, for him

to study long division without first knowing subtraction.

This cumulative nature of mathematics makes it imperative that mathematics teachers consider the important fact of logical sequence.

At times, we hear of conflicts in teaching ideas between what is called the psychological approach and what is called the logical approach. My first-grader son started taking piano lessons a few weeks ago. His teacher uses a so-called psychological approach. He didn't have to learn the scale, the names of all the notes, the symbols for eighth notes, quarter notes, half notes, sharps, flats and all, before playing a tune. In fact, he was elated when he came home from his first lesson and said, "I can play six tunes. I'm really going to practice this week so that I can play them better."

If one thought through to the best psychological way to teach mathematics, one would find that the psychological way of necessity must be a logical way. Mathematics is a body of logic. One idea precedes another in a chain of ideas. A student learns a new idea by going on from old familiar ideas. There is no "Royal Road of Learning Mathematics." This does not imply, however, that there is only one road, but, whatever the road, there is a superstructure of ideas that must be followed.

Since mathematics is learned in this cumulative manner, the procedure the teacher should use is easy to outline, but difficult to execute. The procedure is this: Find out just what the student presently knows and start building on that.

Under the present class organizations, with the great differences in level of achievement that we mentioned, you can see how difficult this is for teachers to do. Some teachers have developed techniques of class organization and procedures to help with this situation, such as grouping within classes, supervised study, differentiated assignments, special projects and so on, but such procedures have not proved adequate to the task. They will prove less adequate, it seems to me, as classrooms become more and more and more crowded, as teachers are given more classes to teach, and as teachers are sometimes hired who are not fully qualified.

One method of dealing with this problem of differences in ability is homogeneous grouping. This method needs much more emphasis than is presently being given it.

Those of us who have studied the history of the development of our elementary schools know that some fifty to seventy-five years ago there was a great emphasis placed on the importance of the graded school. So many schools at that time were the one-room schools. Educators pointed out that if these schools could be organized into graded schools much more efficient instruction would result. Students who were at about the same level of achievement would then be in the same room with the same teacher. The teacher could pitch the instruction to this level.

Over a period of years, the graded schools have become graded principally according to chronological age rather than achievement. As I pointed out earlier, in one Illinois school 97% of the pupils in grade eight were within one year of the median age for that grade. The achievement level in arithmetic was from grade 4 to grade 13, or a span of nine years.

It seems to me that when we consider the way mathematics is learned, because of its cumulative nature, and also consider the terrific teaching problem that results from such a large span, we can be led to one important conclusion. Namely, the administrative organization of the junior high school should make homogeneous groupings on the basis of achievement possible in mathematics.

From experience I know that this is a controversial subject and a subject that will arouse heated debate within a school faculty. For example, we hear arguments presented that this homogeneous grouping is not democratic. Why, I am not sure.

Possibly this is because the opponents believe students are being grouped against their wishes and that it is every student's democratic right to be in group number one.

In my conception of democracy I believe it is every student's right to have the opportunity to learn. Under the present organization of heterogeneous groups, I have seen many students become bored, unchallenged, and lazy because the subject matter was too easy. I have also seen many students in classes I have visited just sit in the class day after day learning only a few mathematical notions because they do not have enough fundamental knowledge in mathematics to follow what is going on. They are not receiving their democratic right to an opportunity to learn. They should be permitted to be in a group nearer their level in mathematics achievement.

Another argument presented against homogeneous grouping is that the slower students ought to be with superior students so that they can be inspired by them. Superior students, who are potential leaders, need to be with slower students whom they may eventually lead in order to understand them better and learn to work with them. With these arguments I have a certain amount of agreement for education generally. However, these arguments are not applicable to mathematics learning specifically.

In our high school at home we have a varsity band, a second band, and other bands. A poor trumpet player is not permitted to be in the varsity band just because it is good for him to be with superior trumpet players or because it is good for the superior trumpet players to learn how to get along with him.

Homogeneous groupings do not need to be made in every subject in school. Possibly the social studies classes, the home rooms, the extracurricular activities, are good places for the superior and the slow to be in the same class.

For mathematics, however, I believe that the achievement level of each student should be measured as well as it can be measured, and that he should be placed in a group for instruction that is approximately at his achievement level.

Another argument that is presented against homogeneous grouping is that it is hard on a student's mental health to know that he is in one of the slower groups, and that no matter how well you disguise the titles of the groups the students will still know what group they are in. I know of no evidence that shows that grouping of students has been the cause of poor mental health. I do not accept this as a valid argument. I am inclined to believe, on the other hand, that students who are so far behind the class that they are constantly failing, may develop a feeling of frustration and defeat that *is* hard on their mental health. There is some evidence available that constant defeat does affect mental health. But success is a wonderful tonic.

The homogeneous grouping of students in mathematics could be made on the basis of *ability* or it could be made on the basis of *achievement*. It seems to me that *achievement-level* division is what is needed. This *achievement-level method*, however, needs to be supplemented by a very carefully prepared testing program that measures achievement. The emphasis upon achievement rather than ability seems to me to be better because: (1) The cumulative nature of mathematics learning requires prerequisite knowledge to new knowledge; and (2) The division of students into achievement groups can be more adequately defended to parents and students than can ability groups.

In the present traditional organization of the junior high school there are only three times that achievement levels relative to passing or failing are checked. These are at the end of grades seven, eight, and nine. If a student fails he has to take an entire year over. This is a terrific price to pay, and many teachers are unwilling to ask many students to do it.

Instead of three achievement periods, grades seven, eight, and nine, I would like to see at least six and possibly twelve achievement periods in the three years of junior-high-school mathematics, with careful measurement of achievement at the end of each period. That is, I would propose an examination and decision at the end of the first nine weeks. Those students who passed would proceed to the next nine weeks' work. Those who failed would meet with another group who were at about the same achievement level. The mathematics classes would be reorganized at the end of every nine weeks. By the end of the first year some students would have advanced four achievement levels, some three, and some only one or two. Some of the students would also have been advanced to groups five or six, because of their superior knowledge in the subject.

I am sure such a proposal as this would increase expenses in teaching mathematics, but I do not believe it is beyond the reach of many communities interested in paying for good education.

The proposal I have made has several advantages, which I believe would more than pay for the added expense:

(1) It makes possible organization of classes that are near the same achievement level. The present program does not efficiently do this.

(2) It places an emphasis on reaching a certain achievement level before a student moves on to the next level. This gives the student more immediate goals, which helps him apply himself to his work.

(3) It means that the instructors will place more emphasis upon measurement and diagnosis of student difficulty.

Possibly this proposal is too much of a dream. I believe, however, that emphasis upon reaching achievement levels, combined with grouping students for instruction according to achievement level would have a very real influence upon improving instruction in mathematics.

Certainly, I do not favor such a program as I have outlined for all subjects. Mathematics, as I pointed out, must be learned in a logical manner. The chain of logic in mathematics is the heart of mathematical meaning. I believe that an instruction program should be planned according to the way students learn.

Many schools use homogeneous grouping plans already, but many schools do not. I have visited two or three schools that divided the students into homogeneous groups, but after doing so made no difference in the instructional materials or the speed of covering the material. If the method is to work, the instruction must be geared to the achievement and ability level.

It is encouraging to note in a recent publication, *A Design for Early Secondary Education in New York State* (Albany: The University of the State of New York, 1954), p. 23, the following statement:

> On the other hand there are certain skills and knowledges which are more readily and more economically acquired in a systematic, cumulative fashion at various rates and which do not require for their achievement direct or extended experience in a diversified social group. For such learnings as reading and mathematics, it is easier to provide individual help and to instruct a group if the pupils do not vary too greatly in their level of competence. Schools sometimes narrow the range of abilities in such classes in order that ideas may be more readily communicated, that pupil's interests will be challenged, that the incentive to learn will be strengthened and that there may be a reasonable equality of competition.

Enrichment and advancement

I believe that the students who need a greater challenge should be accelerated somewhat so that many of the top students will have completed the usual first year of mathematics before the end of the ninth grade. These students, good students that they are, can also be provided with considerable enrichment material in mathematics. For example, they can spend some time on developing skill in

mental arithmetic. I think it is too bad that so many students finish school today dependent on paper and pencil for the most simple computations.

These advanced students could profitably study supplementary topics, such as computing devices: the abacus, the slide rule, or electric calculators. They could study units on the history of number and numbers to bases other than ten. The binary numbers are receiving a lot of attention these days, and their study would be very suitable for advanced students in junior high school.

I believe a unit on the earth as a sphere would be very profitable for junior high schools. I have found in my own teaching that students are fascinated by a study of such ideas as latitude, longitude, distance on a great circle, nautical miles, relationship of time to longitude, the international date line, time zones, and the difference between local apparent time and standard time. I believe that not only is such a unit interesting, but it is extremely valuable in this modern age, where events all around the world are of so much interest to us.

Many other topics from geometry, trigonometry, and logic would make suitable enrichment topics for the above-average junior-high-school students.

Special attention to the gifted

Enrichment in mathematics is not enough for some of the students in the junior high school. Some students at this grade level have the capacity and the interest to advance rapidly in mathematical thinking. Our educational program should provide encouragement and help for these students. At least, the schools should not stand in their way.

If you will read the biographies of famous mathematicians, as many of you have, I am sure you will be struck by the fact that many of these men were famous mathematicians before they were 25. In some cases this was before they were 20. Genius in mathematics frequently develops at a very early age. Junior high school teachers must learn how to recognize the gifted student in mathematics and encourage him through special attention.

This last year I helped with a state contest in secondary school mathematics in Illinois. The top ten students in the state had these scores: 107, 92, 89, 88, 86, 81, 79, 76, 74, and 74, on a very difficult test. Three-fourths of the scores (of the best students in the state) were below 46, and half of the scores were below 32, so you can see that the winners' scores were very exceptional since the items were not easy. An interesting fact is that so many of the top ten scores were made by younger pupils. The boy who made the highest score of 107 was only 16 years of age, and he was competing with others 17 and 18 years of age. The boy who made a score of 76 is in junior high school. By age he is an eighth-grader, but due to an extra promotion he is now in the ninth grade. He already knows high-school mathematics well enough to score eighth in the state of Illinois against juniors and seniors. The boy who was second with a score of 92 studied calculus last year as a junior in high school.

It is urgent, for the student's own good and for the good of society, that such ability in mathematics be recognized and nurtured at the junior-high-school level. In many schools today this is not done.

Problem of qualified teachers

One very real problem in junior-high-school mathematics teaching today is the scarcity of adequately trained mathematics teachers for this age group. I think our record at Illinois is not too different from that of other universities and schools. Our enrollments in the mathematics teacher curriculum have been very low. In the spring of 1953 we had nine graduates in mathematics teaching, secondary level. Of this group three are presently teaching. The others are now housewives, in the armed forces, or attending graduate school.

In 1954 we had eleven graduates. Of this group only three are teaching. Marriage, armed forces, and graduate schools account for the rest.

This year, 1955, we had seven graduates. We are hoping that at least three of these people will go into teaching.

Of course, some of the persons now in the armed forces or going to graduate school will eventually enter teaching, but the attrition rate for the field will be high.

This low number of prospective teachers being graduated occurs at a time when the great population increase is showing up in the junior high school. I am confident that many of our graduates would have made excellent junior-high-school teachers; several of them had their student teaching experience at the junior-high-school level and expressed enjoyment at working with junior-high-school-age students.

At Illinois, we presently do not have a special curriculum for the training of junior-high-school teachers. The teachers are either trained for elementary teaching, with emphasis on grades 1 to 6, or for high-school teaching, with emphasis on grades 9 to 12. The junior-high-school level is a kind of no-man's land as far as teacher preparation is concerned. This situation is not unique to our institution, I am sure.

The junior-high-school level is also a kind of no-man's land as far as certification procedures are concerned. The elementary certificate in Illinois is good for grades 1 to 8. The secondary certificate is good for grades 7 to 12.

An employer can hire either an elementary teacher or a secondary teacher for the junior-high-school mathematics courses. It is possible in Illinois for a teacher to be teaching eighth-grade arithmetic with no high-school mathematics and only one five-hour course in arithmetic in college.

Good administrators in the best schools do not make such teaching assignments, but it is possible to do, and it is done. It will be done much more in the years to come when the large school populations and the small number of graduates trained to teach mathematics coincide.

Recently, an all-university committee was appointed at Illinois to give special consideration to the problem of training and certifying junior-high-school teachers. A very few schools in the United States do have special programs for these teachers, but the vast number do not. I do not know what is the best way to accomplish the needed improvement, but I am firmly convinced that a tightening of certification requirements for junior high schools is needed.

To be realistic, it would be some improvement if we could get the standards raised to require all junior-high-school mathematics teachers to have at least twelve hours of college mathematics. This would be better than the five now required in Illinois. At the University of Illinois we require thirty-two hours of mathematics in our curriculum for preparation of high-school mathematics teachers. I would not want to lower this. I think teachers in the upper grades of high-school mathematics programs need this much work in mathematics. However, to hold to this same standard for junior high schools is not realistic, especially since the junior-high-school teachers can come in by way of a minor in mathematics or by way of elementary certification with only one five-hour course in the teaching of arithmetic.

The training of an adequate supply of qualified mathematics teachers is certainly one of the major problems in mathematics education today.

The junior-high-school years are extremely important for the development of interests. A good mathematics teacher, who enjoys mathematics, teaches it well, and enables students to achieve success, can do much to inspire more students to enroll in high-school mathematics courses and thereafter maintain a life-long interest in mathematics.

Problems in mathematics teaching related to junior-high-school curriculum trends

In recent years there has been a significant number of junior high schools that have been experimenting with various types of curriculum patterns that cut across subject boundaries. These experiments have been given various names such as "common learnings," "core curriculum," "unified studies," and "block-of-time programs." These curriculum patterns have been advocated and tried in various degrees. They have been advocated because many teachers and educators claim that students at the junior-high-school level should get some experience in problem-solving and should tackle problems significant to them as students. These problems may be of personal, school, or community interest. Actual problems are not limited to only one subject. Consequently, these educators say the student should have practice solving these real problems, and in the process he will learn something about social studies, English composition, and so on.

The question we need concern ourselves with here is: "What is the place of mathematics learning in such curriculum plans as those mentioned?"

In my opinion, such a plan as a core curriculum is not sufficient for mathematics learning. The very nature of mathematics learning makes the task impossible. As we pointed out earlier in this paper, mathematics is a cumulative subject, where a new idea is learned from previous ideas. If the previous ideas are not present, the new ideas are not possible. This indicates, it seems to me, that a separate time of day should be set aside for mathematics learning. Mathematics cannot be studied unless it is studied as a system.

Advocates of the various types of integrated curriculum plans at the junior-high-school level offer various criticisms of the present organization of courses. I should like to mention two of them.

(1) A teacher in a departmentalized junior high school has too many students to become well enough acquainted with them to be of maximum help. It would be better if the teacher could have fewer classes, so that he could take a more direct interest in each student. The teacher would also feel a greater responsibility for the students because they would then be thought of as *his* students, rather than just the group of pupils that come to see him for forty minutes a day.

This argument makes sense, too. Not long ago I visited an eighth-grade mathematics class at a junior high school. The teacher had every one of the thirty-six chairs in the room filled. A student had to go out for a chair for me, and I had trouble finding a place to put the chair, the room was so crowded. This teacher told me he had seven classes in eighth-grade arithmetic a day. They were all about the same size. I do not know how this teacher could be of help to his students with their individual difficulties. In addition to great differences in ability and achievement in mathematics, these students also had many social and emotional problems as young adolescents often have. Frequently these emotional and adjustment problems are barriers to learning and need attention. A teacher in a situation such as I have described, however, has little opportunity to help.

So you see, I am in agreement with the basic idea that a junior-high-school student should be with one teacher for more than one period if possible. However, if this means that he must be with a *weak* teacher for two or more periods, or a teacher who doesn't know mathematics, then I am not for it. A little while ago we were discussing the shortage of qualified mathematics teachers at the junior-high-school level. If this shortage continues, it may be impossible or inadvisable to use qualified mathematics teachers for any

other subject than mathematics. This would make some of these curriculum plans impossible.

(2) A second argument that critics of the present arrangement of courses make is that highly departmentalized programs tend to be taught independently of each other. This procedure gives very little opportunity for correlating learning in one subject with learning in another. Under a well qualified mathematics and science teacher both the mathematics and the science could be supplemented and helped by the study in the other subject.

I taught in such an arrangement once. Although most of the time I taught mathematics in the mathematics class and science in the science class, several times mathematics was taught during the science hour and science during the mathematics hour. I believe the students learned better in both subjects because of this arrangement.

Again the key to the plan is the availability of qualified teachers.

Some teachers in a junior high school told me their school had started a block-of-time program. This resulted in the need to press into service a few teachers in other subjects to teach some mathematics. A few of these teachers openly stated that they had never taught mathematics and had very little interest or training in the subject. One teacher said, "I hate arithmetic, but if I have to teach it I will." This is *tragic*. The block-of-time idea has its good points, but these advantages are greatly dependent upon qualified, interested teachers.

In a teacher shortage situation I believe the departmentalized junior high school, as far as mathematics is concerned, may be a necessity for adequate mathematics instruction.

Mathematics content

I have not omitted a discussion of the mathematics content at the junior-high-school level because I did not believe it important. I do not believe, however, that at the present time the problems relative to content are as critical as the problems I have mentioned here—individual differences, adequate supply of qualified teachers, and adequate facilities. These are the really critical problems which demand solution if mathematics teaching at the junior-high-school level is to improve.

Chiquita Trigonometry

I'm Chiquita Trigonometry, and I've come to say:
Radicals have to be treated in a certain way.
When they are cube root and square root, too,
Radicals on the top are the best for you.

You can put them on the top.
You can put them on the side.
Anywhere you want to place them,
It's so easy to erase them.

For radicals like the climate
Of the very, very tropical numerator.
So don't put your radicals in the denominator.
No, no, no, no!

—by Bernie McDyer and Jerry Silverman,
Students, Henry Snyder High School,
Jersey City, New Jersey

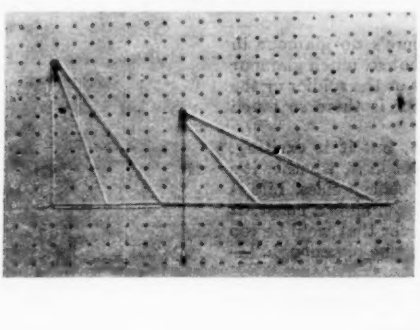
Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota

A multi-model demonstration board

by Donovan A. Johnson, University of Minnesota, Minneapolis, Minnesota

A versatile demonstration board can be yours for less than $2. And better yet no construction work is required. Buy a piece of hookboard 30 inches square from your local lumberyard or mail-order house. It will cost about 20 cents per square foot. These hookboards have evenly spaced holes in both rows and columns at one-inch intervals. Masking tape may be used to cover the edges and give the appearance of a frame. Painting the board with blackboard paint makes it possible for one to produce chalk lines on its surface. However, this is optional; lines may also be produced with colored elastic thread, colored cellophane tape, or brush pen, and the surfaces of the board painted in colors other than black. Elastic thread is the best material for representing lines which change position. Hooks or pegs which fit into the holes are available for holding the thread in place. The elastic may also be threaded through the hole if desired. Threads may be fastened behind the board with knots or by means of pegs drawn through loops. A circular protractor with a hole at its center large enough to accommodate a peg completes the demonstration board. Composition acoustical tiles 12 inches square may be used individually by students to complement the use of the large demonstration board suggested for the teacher. Both types of boards may be used to develop and illustrate a variety of geometrical ideas and theorems like the following:

Figure 1



Meaning and measurement of a plane angle. Attach the protractor at any convenient point with a peg. With elastic thread form angles having their vertices at the center of the protractor. Measure the angles and show that the size of an angle is independent of the length of the sides of the angle. Show that vertical angles are equal.

Different kinds of polygons. Polygons of any number of sides may be formed by stretching elastic between points locating the vertices. The protractor may be used to measure interior and exterior angles.

Areas of polygons. An approximate value for the area of any polygon can be found by counting the one-inch squares enclosed by the thread which outlines its perimeter. The areas of rectangles and parallelograms with equal bases and equal altitudes may be easily compared with the

aid of the device. Also, the equality of triangles having constant bases and constant altitudes may be illustrated with ease.

Secondary lines of polygons. Form triangles of different sizes and shapes. "Stretch in" the altitudes, perpendicular bisectors, angle bisectors, and medians to show the concurrency of these lines (Figure 1). Show that similar polygons can be divided into similar triangles that are similarly placed.

Relationships of lines of quadrilaterals. Form quadrilaterals of varied shapes and sizes including parallelograms and trapezoids. Show the relationships between diagonals. Connect the midpoints of sides to show that the figures formed are all parallelograms.

The Pythagorean theorem. Form a right triangle. Outline the square on each side. Compare the areas of the squares to derive the formula $c^2 = a^2 + b^2$.

The base angles of an isosceles triangle are equal. Form isosceles triangles of varied sizes. Measure and compare the base angles of these triangles by using the protractor.

Parallel lines. By using elastic to represent a transversal, pairs of equal angles may be illustrated. That parallel lines are everywhere equidistant may also be illustrated.

Vectors. Use the elastic thread to illustrate vectors and the concept of vector addition.

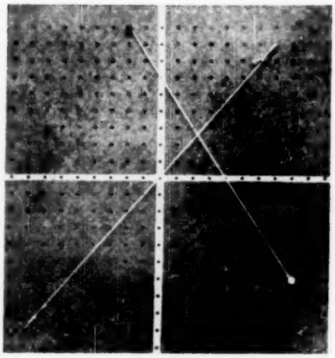


Figure 2

Circle relationships. Using string or a strip of cardboard draw a circle with chalk. Use elastic thread to give meaning to the definitions of such terms as diameter, radius, chord, inscribed angle, central angle, and inscribed polygon.

Graphing. Form rectangular coordinate axes with colored cellophane tape. Label the axes with an appropriate scale. Points may be plotted by placing pegs in holes. Elastic thread attached to pegs may be thought of as representing the graph of an equation or a broken line graph. Bar graphs or histograms can be formed by pegging strips of colored cardboard to the board (see Figure 2).

"Perhaps you think there's no glamour in teaching. Perhaps there's not so much glamour as in the life of the explorer or the soldier or the Hollywood star. But neither is there so much drudgery or mud.

"Teaching isn't all fun, but it is exciting. There's never a dull moment in it, whether it's in the kindergarten or the college class. Life is dull only to dull people. Teaching is a constant challenge. The danger is never that we'll be too big for it, only that we won't be big enough."—*James M. Spinning*

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Irrationals or incommensurables I: their discovery, and a "logical scandal"

by Phillip S. Jones

The story of the earliest beginnings of the idea of incommensurable quantities is so uncertain as to be still the basis of lengthy scholarly discussions airing somewhat inconsistent scholarly opinions. No one *knows* exactly who first discovered two incommensurable quantities, when this was done, how the discovery was made, or what was the motive or interest which led to the discovery.[1] However, in spite of the lack of agreement on details, enough of the story is known and enough agreement does exist to reveal rather clearly the development of mathematical thought and Greek motives at an interesting period in the history of the world as well as to demonstrate still usable approaches to the fascinating and conceptually difficult topic of irrational or incommensurable numbers.

To put the Greek work on incommensurables into proper perspective, one should first note that two problems which we today recognize as involving irrationals had occupied important positions in Egyptian and Babylonian mathematics without apparently, however, stimulating in their minds any thought of the concepts which are our topic. These are the problems of the circumference and area of circles in connection with which we today think of the number π which is not only irrational but also transcendental, and the problems involving the Pythagorean theorem and Pythagorean triples.[2] The circle problems do not seem to have brought either Egyptians or Babylonians at all close to the concept of irrationals. However, the Babylonian use of the Pythagorean theorem in situations where the hypotenuse was irrational and their approximations to the square roots involved in these cases may be regarded as steps toward a discovery which they never made. Further, the recently dis-

[1] Some of the differences in opinion are revealed in the content and footnotes of the most extensive recent discussion by Kurt von Fritz, "The Discovery of Incommensurability by Hippasus of Metapontum," *Annals of Mathematics*, Vol. 46 (April 1945), pp. 242–264. Other basic sources which give further data and references are Sir T. L. Heath, *History of Greek Mathematics* (Oxford: The Clarendon Press, 1921) and Sir T. L. Heath, *The Thirteen Books of Euclid's Elements* (Cambridge: The University Press, 1908).

[2] For discussions of the Egyptians' procedure for finding the area of a circle and more recent facts about π see THE MATHEMATICS TEACHER, XLIII (March 1950), pp. 120-121, and (May 1950), p. 208; N. T. Gridgeman, "Circumetrics," *The Scientific Monthly*, LXXVII (July 1953), p. 31 ff. as well as the standard histories of mathematics, of course.

For discussions of the Pythagorean theorem, Pythagorean triples, and Babylonian extractions of square root, see especially THE MATHEMATICS TEACHER, XLII (October 1949), pp. 307-308, and XLIII (April 1950), pp. 162-163; as well as XLIII (October 1950) p. 278; XLIII (November 1950), p. 352; XLIV (April 1951), p. 264; XLIV (October 1951), p. 396; XLV (April 1952), p. 268; XLVI (March 1953), p. 188; XLVI (April 1954), p. 269.

covered Babylonian tables of Pythagorean triples preceded and could have stimulated the later Greek study of these triples and of figurate numbers. Both of these topics can be associated with the Greek discovery of irrationals.

Although modern scholars varyingly locate the discovery of the incommensurables from 450 B.C. to 375 B.C., they all agree that it took place after Pythagoras (540 B.C.) and before Plato (430?–349? B.C.) in spite of Proclus (450 A.D.) who wrote, "Pythagoras transformed this science [geometry] into a free form of education; he examined this discipline from its first principles and he endeavored to study the propositions, without concrete representation, by purely logical thinking. He also discovered the theory of irrationals (or of proportions) and the constructions of the cosmic solids (regular polyhedra)."[3] Fritz's recent study supports the ancient legends that it was Hippasus of Metapontum (*circa* 500 B.C.) who first *discovered* the existence of incommensurables. The legends also relate that he was later lost at sea as punishment presumably for having revealed his discovery.

To understand either the ancient meaning of this legend or more recent interpretations, one must recall that the Pythagorean brotherhood was actually a somewhat aristocratic group of people carrying on political as well as philosophical-mathematical activities in Italy where Pythagoras himself had lived at Crotona. These followers of Pythagoras, whose recognition symbol was the mystic pentagram formed by extending the sides of a regular pentagon, were finally broken up and driven out of Italy after the rise of Athenian democracy which followed the defeat of the Persians by Athens and Sparta. Pythagorean philosophy had two interrelated aspects, one mystical-religious, and the other scientific-mathematical. B. L. van der Waerden's summary of this is:

> Their doctrine proclaims that God has ordered this universe by means of numbers. God is unity, and the world is plurality, and it consists of contrasting elements. It is harmony which restores unity to the contrasting parts and moulds them into a cosmos. Harmony is divine, it consists of numerical ratios. Whosoever acquires full understanding of this number-harmony, he becomes himself divine and immortal. Music, harmony, and numbers—these three are indissolubly united according to the doctrine of the Pythagoreans.[4]

From this you may see why anything which would upset their theory of numbers would be most distressing to the Pythagoreans. Since their theory of numbers was essentially the theory of integers and the ratios of integers, the discovery of incommensurable geometric magnitudes, that is, magnitudes whose ratio could not be represented by pairs of integers, was to them what one modern writer has called a "logical scandal." Further, in its earliest days Pythagorean doctrine was to be transmitted only by word of mouth and the works of his followers were credited to the master; hence, when Hippasus not only revealed a Pythagorean discovery, but also revealed such an upsetting one, he certainly made himself a fair target for the wrath of his fellows if not of the gods.

Van der Waerden reads a much more modern situation into the Hippasus legend. He sees the existence of a conflict between those followers who after his death wished to regard Pythagoras' teachings as the source of all knowledge and those who felt one must rely on one's own thoughts in the search for truth. He believes that science cannot advance if it is a secret known only to initiates, that scientists will inevitably come into conflict with a pledge of silence, and that Hippasus' expulsion from the Pythagoreans came because he deliberately both added to and revealed Pythagorean doctrines.

[3] B. L. van der Waerden, *Science Awakening* (Groningen, Holland: P. Noordhoff Ltd., 1954), p. 90, which quotes from Proclus' commentary on Euclid's *Elements* as reported by Friedlein.

[4] van der Waerden, *op. cit.*, p. 93.

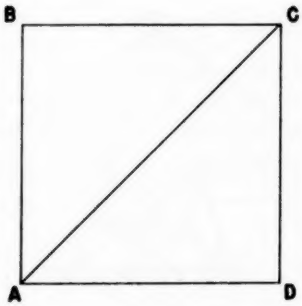


Figure 1

So far we have discussed some of the historical arguments and legends relating to the *discovery* of the existence of incommensurables. A related question is: How were given pairs of quantities first *proved* to be incommensurable? Although one could argue that nothing, especially in mathematics, is discovered until it has been proven it is also important for teachers and students to recognize that the discovery or formulation of a conjecture is often as important, and sometimes more inspired, than the proof which must be worked out before the conjecture is fully established as a valid theorem.

The oldest definitely known *proof* associated with incommensurables corresponds in its essential elements to the modern proof that $\sqrt{2}$ is irrational. This is the proof that the diagonal and side of a square are incommensurable and is to be found as an addition, probably by Theaetetus, at the end of Euclid's Book X. In spite of its geometric formulation the proof there depends not only on the Pythagorean theorem (which probably was not proven until after Pythagoras!), but also on the Pythagorean theory of even and odd numbers which does go back to the time of Pythagoras. Euclid expounded this latter theory in Book IX of the *Elements*. The essential facts of number theory needed are that an odd number times an odd number is odd and an even times an even is even. From this, since all integers are even or odd, it follows that if a perfect square is even, its square root is also even. The indirect method of proof was used. The negative of the desired conclusion was assumed and a contradiction was derived from this assumption. In condensed form the proof was: Assume that in square $ABCD$ in Figure 1, AC and AB are commensurable, i.e., $AC:AB=m:n$ where m and n are integers with no common factor, that is, not both m and n are even. By the Pythagorean theorem $AC^2=2AB^2$, whence $AC^2:AB^2=2=m^2:n^2$. From this $m^2=2n^2$. Thus since m^2 is even, m is also even, or $m=2p$. Thus, since m is even, n is odd. Substituting for m and simplifying we have $2p^2=n^2$ which shows n also to be even. Thus, we have shown the same number, n, to be both even and odd, which is impossible. Hence, the original assumption that AC and AB are commensurable is false.

The earlier study of Pythagorean triples and of figurate or polygonal numbers may well have paved the way for such a proof by associating the theory of integers with geometric figures and by stimulating mathematicians to search for common elements, such as ratios, possessed by a class of figures such as similar triangles or right triangles. However, though this may have been the first proof that a pair of lines were incommensurable, Kurt von Fritz believes that the original *discovery* of the existence of incommensurables was due to Hippasus and grew out of a study of the pentagon rather than the square. This conjecture is based on the historical facts that the pentagon and pentagram were important Pythagorean symbols, that Hippasus was also concerned with the pentagon when he solved the problem of inscribing a regular dodecahedron in a sphere, and the psychological fact that the incommensurability of the side and diagonal of a pentagon is almost visually evident as shown in Figure 2.

Further, the chief theorem needed to construct a proof of incommensurability in this case is the isosceles triangle theorem associated with Thales, the first known

Greek geometer. The only other concepts needed are that angles inscribed in equal arcs are equal and the method of successive subtractions which was also a common tool in Greek mathematics. The gist of the proof is to show that in the successive smaller and smaller pentagons formed by repeatedly drawing diagonals as shown in Figure 2 a common measure of the side and diagonal of one pentagon would also be a common measure of the side and diagonal of each smaller pentagon. Since these approach zero as a limit, the common measure is zero or does not exist.

The proof for all of this follows from observing that all the angles at each vertex of $ABCDE$ are equal and that $\angle BAC'$ is equal to $\angle BC'A$. From this, $AB = BC'$ and

(1) $BE - BA = C'E = EB' = B'D' = BD'$.

(2) $AB - BD' = BC' - BD' = C'D'$.

(3) $B'D' - C'D' = E''B' = A''D''$.

If we symbolize successive sides and diagonals as follows: $AB = s_1$, $C'D' = s_2$, $A''E'' = s_3$, and $BE = d_1$, $B'D' = d_2$, $A''D'' = d_3$, we then have

(1) $d_1 - s_1 = d_2$

(2) $s_1 - d_2 = s_2$

(3) $d_2 - s_2 = d_3$

(4) $s_2 - d_3 = s_3$, etc.

From (1) we conclude that a common measure (divisor) of d_1 and s_1 will also measure d_2. From (2) and (3) we see that such a common measure will also measure s_2 and d_3, etc. This intuitively completes the proof that there is no common measure, or that the side and diagonal of the pentagon are incommensurable, because we see that the assumption that there is such a (finite) common measure leads to the contradiction that it must be zero. A rigorous proof requires either a more modern idea of limit or the use of some form of the axiom of Archimedes. This axiom, which really antedates Archimedes, says that if from any quantitity we take away its half or more, and then from the remainder we take away its half or more, and

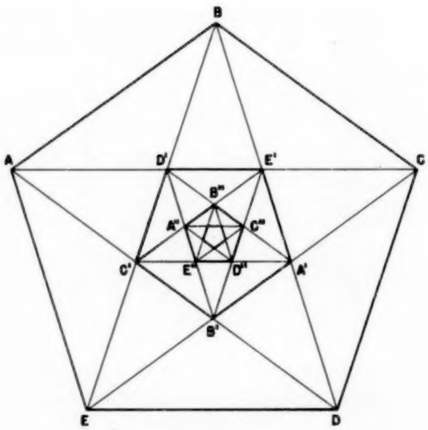


Figure 2

so on, we can ultimately reach a remainder which will be less than any previously given or assumed fixed quantity.[5] This axiom is, of course, closely allied to the limit concept, which is fundamental in any modern treatment of irrationals.

Whether or not Hippasus *discovered* the existence of irrationals, and if so whether via studying the mystic pentagon or pentagram or a square, whether or not he *proved* this existence and if so whether via successive subtractions or the Pythagorean theory of even and odd numbers may never actually be known. However, these conclusions all seem valid:

1. The *discovery* and *proof* both took place in the period 450 B.C. to 375 B.C.
2. Both the pentagon and the square were the subject of study at that time.
3. Both the methods of successive subtractions and of even and odd numbers were known at about this period, were being used, and were applicable to this problem, although the latter is the method which appears in the earliest discussions of incommensurables which we now have.
4. All of these are reasonably understandable, concrete, and elementary ap-

[5] This statement is actually Euclid X.1. which he proved by use of the more generally recognized form of Archimedes' Axiom. See Sir T. L. Heath, *Manual of Greek Mathematics* (Oxford: The Clarendon Press, 1931).

proaches to a problem which is still of importance in mathematics.

In our next installment of this story of irrational numbers, we will see how the method of successive subtractions could be applied to the side and diagonal of a square and how this associates with a method of finding rational approximations to the $\sqrt[2]{2}$. After this we will present the Greek solution to the dilemma of the incommensurable, namely, Eudoxus' method of exhaustion.

What's new?

BOOKS

College

A Manual of Problems in Statistics, rev. ed., Scott Dayton, New York, Henry Holt and Company, 1955. Paper, v+137 pp., $1.95.

Basic Mathematics for Science and Engineering, Paul G. Andres, Hugh J. Miser, Haim Reingold, New York, John Wiley and Sons, Inc., 1955. Cloth, vii+846 pp., $6.75.

Elementary Statistics, R. Clay Sprowls, New York, McGraw-Hill Book Company, Inc., 1955. Cloth, xiii+392 pp., $5.50.

Miscellaneous

Shop Mathematics, Claude E. Stout, New York, John Wiley and Sons, Inc., 1955. Cloth, xi +282 pp., $3.70.

The Bequest of the Greeks, Tobias Dantzig, New York, Charles Scribner's Sons, 1955. Cloth, 191 pp., $3.95.

Your Life Plans and the Armed Forces, Textbook, Defense Committee of the North Central Association of Colleges and Secondary Schools, Washington, D. C., American Council on Education, 1955. Paper, x+149 pp., $2.00.

Your Life Plans and the Armed Forces, Teacher's Handbook, Defense Committee of the North Central Association of Colleges and Secondary Schools, Washington, D. C., American Council on Education, 1955. Paper, v+23 pp., $.60.

BOOKLET

Math Problems From Industry, Educational Services, Department of Public Relations, Chrysler Corporation, P.O. Box 1919, Detroit 31, Michigan. Illustrated booklet giving 51 problems; solution book available to teachers; free for limited number of classroom sets requested on school stationery.

DEVICES

Combination Drawing Instrument (No. 7551), W. M. Welch Manufacturing Company, 1515 Sedgwick Street, Chicago 10, Illinois. Plastic and metal instrument, $3.50 each.

C-Thru Architects' Scalemaster (No. AR-46)

C-Thru Engineers' Scalemaster (No. EN-56)
C-Thru Ruler Company, 827 Windsor Street, Hartford, Connecticut. Plastic rulers, $3\frac{3}{8}'' \times 6\frac{3}{4}''$, with 14 and 9 scales respectively; list price for either, $7.20 per dozen.

Pattern Dial (No. 7552), W. M. Welch Manufacturing Company, 1515 Sedgwick Street, Chicago 10, Illinois. Plastic and metal instrument, $3.50 each.

EQUIPMENT

Acu-Math Student Mannheim Slide Rule No. 400, Yoder Instruments, 142 North Market Street, East Palestine, Ohio. 10" plastic slide rule with leatherette sheath and instruction book, $2.00 each, quantity discounts.

C-Thru 10" Slide Rule (Cat. No. 88), C-Thru Ruler Company, 827 Windsor Street, Hartford, Connecticut. Plastic slide rule with case and instructions; list price, $1.80 each.

Slated Globe, Cradle Mount (No. 00221), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. 12" metal slated globe in metal cradle, with equatorial ring, $27.50.

FILMS

"Understanding Numbers Series":
The Earliest Numbers (#IS-233)
Base and Place (#IS-234)
Big Numbers (#IS-235)
Fundamental Operations (#IS-236)
Short Cuts (#IS-237)
Fractions (#IS-238)
New Numbers (#IS-239)
Audio-Visual Center, Indiana University, Bloomington, Indiana. 16 mm. films, 30 minutes each, B & W. Originally produced for the Educational Television and Radio Center of Ann Arbor, Michigan. Each film features Phillip S. Jones. Purchase price of each film, $100; rental fee for each film, $3.50.

MODEL

Demonstration Micrometer Caliper (No. 51), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Blown-up model of micrometer caliper, $65.00.

Edited by Adrian Struyk, Clifton High School, Clifton, New Jersey

A table of Stirling numbers of the second kind, and of exponential numbers

by Francis L. Miksa, Aurora, Illinois

The numbers known as "Stirling numbers of the second kind" have been symbolized in a variety of notations by several different writers. The symbol to be used here is ${}_rS_n$, where r and n are positive integers such that r does not exceed n. Besides being interesting for their own sake because of many intriguing properties, the Stirling II numbers are useful in the calculus of finite differences, in number theory, in summation of series, in problems of enumeration, and in other fields. Because of the widening area of application of these numbers, the writer decided to compute an extensive table of their values. The tabulation was completed to fifty columns, twenty-seven of which are presented here.

The Stirling numbers of both the first and second kinds arise in formulating the relationship between algebraic factorials and powers. Since this is perhaps the simplest application of these numbers, we shall use it to introduce the numbers themselves.

In the customary notation for algebraic factorials we have

$$x^{(1)}=x,\ x^{(2)}=x(x-1),$$
$$x^{(3)}=x(x-1)(x-2),$$
$$x^{(4)}=x(x-1)(x-2)(x-3),\ \text{etc.}$$

In general

$$x^{(n)}=x(x-1)(x-2)\cdot\cdots\cdot(x-n+1).$$

Expanding the indicated products we get

$$x^{(1)}=x,\ x^{(2)}=x^2-x,$$
$$x^{(3)}=x^3-3x^2+2x,$$
$$x^{(4)}=x^4-6x^3+11x^2-6x,\ \text{etc.}$$

Here the factorials have been expressed in terms of the powers. In these expansions the coefficients of the powers are the *Stirling numbers of the first kind.* Let us now invert these results and express each power in terms of factorials. Thus

$$x=x^{(1)},\ x^2=x^{(2)}+x^{(1)},$$
$$x^3=x^{(3)}+3x^{(2)}+x^{(1)},$$
$$x^4=x^{(4)}+6x^{(3)}+7x^{(2)}+x^{(1)},\ \text{etc.}$$

The coefficients in these factorial series are the *Stirling numbers of the second kind.* For this particular application, then, ${}_rS_n$ denotes the coefficient of $x^{(r)}$ in the factorial series for x^n. Employing this notation, we may write as examples of particular cases

$$x^3={}_3S_3x^{(3)}+{}_2S_3x^{(2)}+{}_1S_3x^{(1)},$$
$$x^4={}_4S_4x^{(4)}+{}_3S_4x^{(3)}+{}_2S_4x^{(2)}+{}_1S_4x^{(1)},\ \text{etc.}$$

In general

$$(1)\qquad x^{(n)} = \sum_{r=1}^{n} {}_rS_nx^{(r)}.$$

The accompanying table lists ${}_rS_n$ in the nth column and rth row. Hence for any integer n that does not exceed 27 a particular instance of (1) may be written by

choosing the coefficients from the nth column of the table. Starting at the foot of the column and working upwards insures that successive factorials will be arranged in descending order.

Further instances of the occurrence and use of the Stirling II numbers may be found in the references at the end of this introduction.

To compute the table the author used the fundamental recursion formula

$$ {}_rS_{n+1} = r \cdot {}_rS_n + {}_{r-1}S_n \tag{2} $$

where ${}_0S_n = 0$ and ${}_1S_n = 1$ for all n. To exemplify this and other relationships noted below we reproduce here a few columns of the main table:

$r \backslash n$	3	4	5	6
1	1	1	1	1
2	3	7	15	31
3	1	6	25	90
4		1	10	65
5			1	15
6				1
G_n	5	15	52	203

Since ${}_1S_n = 1$ for all n, each column begins with 1. For column 5, taking $n=4$ in (2),

$$ {}_2S_5 = 2 \cdot 7 + 1 = 15, $$
$$ {}_3S_5 = 3 \cdot 6 + 7 = 25, $$
$$ {}_4S_5 = 4 \cdot 1 + 6 = 10, $$
$$ {}_5S_5 = 5 \cdot 0 + 1 = 1. $$

Each column of the table was checked by means of the equation

$$ \sum_{r=1}^{n+1} {}_rS_{n+1} = \sum_{r=1}^{n} (r+1)\,{}_rS_n. \tag{3} $$

For example, in column 5 the sum of the numbers is 52. Hence, by taking $n=4$ in (3) we should have as a check

$$ 52 = 2 \cdot 1 + 3 \cdot 7 + 4 \cdot 6 + 5 \cdot 1 $$

where the successive entries 1, 7, 6, 1 in column 4 are multiplied by 2, 3, 4, 5 respectively, and the products summed. Similarly, comparing columns 6 and 5, the check is

$$ 203 = 2 \cdot 1 + 3 \cdot 15 + 4 \cdot 25 + 5 \cdot 10 + 6 \cdot 1. $$

Various investigators have shown that the sums of Stirling II numbers for fixed n furnish solutions for several notable problems. This sum, denoted here by G_n, is included in the present table at the foot of each column. Thus

$$ G_n = \sum_{r=1}^{n} {}_rS_n. \tag{4} $$

G_n is also the coefficient of $x^n/n!$ in the Maclaurin expansion of e^{e^x-1}. For this reason these sums are sometimes referred to as *exponential numbers* and are so designated here.

As examples of its association with enumeration problems we mention that G_n is the number of different ways in which

(a) The product of n distinct primes can be factored,
(b) a stanza of n lines can be rhymed,
(c) n distinguishable objects can be put into n or fewer indistinguishable boxes.

In the matter of computation, the values of G_n can be obtained directly, without recourse to the Stirling numbers, by making use of a fundamental difference property. Consider the following array, in which the first row lists the values of G_n in order, and each succeeding row lists the differences of the preceding row. Note that the G_n again appear as the

```
1    2    5    15    52    203
   1    3    10    37    151
      2    7    27    114
         5    20    87
           15    67
              52
```

initial entry in each of the difference-rows. Thus we have

$$ \Delta^1 G_1 = G_1,\ \Delta^2 G_1 = G_2,\ \Delta^3 G_1 = G_3, \text{ etc.,} $$

illustrating the general property

(5) $$\Delta^n G_1 = G_n.$$

Because of this, the G_n can be found as in the scheme below. Begin by writing 1, 1 on the first line, and write the sum 2 below them. Continue the first line with 2, and

$$\begin{array}{ccccccccccccc}
1 & & 1 & & 2 & & 5 & & 15 & & 52 & & 203 \\
& 2 & & 3 & & 7 & & 20 & & 67 & & & \\
& & 5 & & 10 & & 27 & & 87 & & & & \\
& & & 15 & & 37 & & 114 & & & & & \\
& & & & 52 & & 151 & & & & & & \\
& & & & & 203 & & & & & & &
\end{array}$$

then write successively the sum $3=1+2$ on the second line and the sum $5=2+3$ on the third line. Continue the first line with 5, and write 7, 10, 15 as successive sums. Then 15 follows 5 on the first line, and so the process continues.

A relationship useful for testing a table of G_n is the interesting arithmetical property

(6) $$G_{p+k} \equiv G_k + G_{k+1} \pmod{p}$$

where p is a prime. Hence to check $G_6 = 203$, take $p=3$ in (6). Then we should have

$$G_{3+3} \equiv G_3 + G_4 \equiv 5 + 15 \pmod{3}.$$

And, indeed,

$$203 \equiv 20 \pmod{3}.$$

Similarly, $p=5$ and $k=1$ are alternative substitutions in (6) for testing G_6. With these the check is

$$203 \equiv 3 \pmod{5}.$$

A reader who desires to pursue the subject will find interesting accounts and many additional references in the papers and books in the bibliography listed below.

Bibliography

Becker, H. W. "Solution of Problem E 461," *The American Mathematical Monthly,* XLVIII (December, 1941), 701–703.

Becker, H. W., and Riordan, J. "The Arithmetic of Bell and Stirling Numbers," *American Journal of Mathematics,* LXX (April, 1945), 385–394.

Boole, G. *A Treatise on the Calculus of Finite Differences.* New York: G. E. Stechert & Co., 1946. Pp. 11, 18–32, 115.

Ginsburg, J. "Iterated Exponentials," *Scripta Mathematica,* XI (Sept.–Dec., 1945), 340–353.

Jordan, C. *Calculus of Finite Differences.* New York: Chelsea Publishing Co., 1950. Chapter 4.

Moser, L., and Wyman, M. "On an Array of Aitkin," *Transactions of the Royal Society of Canada,* XLVIII (June, 1954), 31–37.

Riordan, J. "The Number of Impedances of an n Terminal Network," *Bell System Technical Journal,* XVIII (April, 1939), 300–314.

Salzer, H. E. Announcement concerning an unpublished "Table of Coefficients for Expressing Powers in Terms of Factorials," *Mathematical Tables and Other Aids to Computation,* I (October, 1944), 330.

Williams, G. T. "Numbers Generated by the Function e^{e^x-1}," *The American Mathematical Monthly,* LII (June-July, 1945), 323–327.

Table of Stirling II Numbers and Exponential Numbers

$_rS_n$ and G_n for $n = 1(1)27$

r \ n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511
3			1	6	25	90	301	966	3025	9330
4				1	10	65	350	1701	7770	34105
5					1	15	140	1050	6951	42525
6						1	21	266	2646	22827
7							1	28	462	5880
8								1	36	750
9									1	45
10										1
G_n	1	2	5	15	52	203	877	4140	21147	1 15975

r \ n	11	12	13	14	15
1	1	1	1	1	1
2	1023	2047	4095	8191	16383
3	28501	86526	2 61625	7 88970	23 75101
4	1 45750	6 11501	25 32530	103 91745	423 55950
5	2 46730	13 79400	75 08501	400 75035	2107 66920
6	1 79487	13 23652	93 21312	634 36373	4206 93273
7	63987	6 27396	57 15424	493 29280	4087 41333
8	11880	1 59027	18 99612	209 12320	2166 27840
9	1155	22275	3 59502	51 35130	671 28490
10	55	1705	39325	7 52752	126 62650
11	1	66	2431	66066	14 79478
12		1	78	3367	1 06470
13			1	91	4550
14				1	105
15					1
G_n	6 78570	42 13597	276 44437	1908 99322	13829 58545

r \ n	16	17	18	19
1	1	1	1	1
2	32767	65535	1 31071	2 62143
3	71 41686	214 57825	644 39010	1934 48101
4	1717 98901	6943 37290	27988 06985	1 12596 66950
5	10961 90550	56527 51651	2 89580 95545	14 75892 84710
6	27349 26558	1 75057 49898	11 06872 51039	69 30816 01779
7	32818 82604	2 57081 04786	19 74624 83400	149 29246 34839
8	21417 64053	2 04159 95028	18 90360 65010	170 97510 03480
9	8207 84250	95288 22303	10 61753 95755	114 46146 26805
10	1937 54990	27583 34150	3 71121 63803	47 72970 33785
11	289 36908	5120 60978	83910 04908	12 94132 17791
12	27 57118	620 22324	12563 28866	2 34669 51300
13	1 65620	49 10178	1258 54638	28924 39160
14	6020	2 49900	84 08778	2435 77530
15	120	7820	3 67200	139 16778
16	1	136	9996	5 27136
17		1	153	12597
18			1	171
19				1
G_n	1 04801 42147	8 28648 69804	68 20768 06159	583 27422 05057

$r \backslash n$	20	21	22
1	1	1	1
2	5 24287	10 48575	20 97151
3	5806 06446	17423 43625	52280 79450
4	4 52321 15901	18 15090 70050	72 77786 23825
5	74 92060 90500	379 12625 68401	1913 78219 12055
6	430 60788 95384	2658 56794 62804	16330 53393 45225
7	1114 35540 45652	8231 09572 14948	60276 23799 67440
8	1517 09326 62679	13251 10153 47084	1 14239 90799 91620
9	1201 12826 44725	12327 24764 65204	1 24196 33035 33920
10	591 75849 64655	7118 71322 91275	83514 37993 77954
11	190 08424 29486	2682 68516 89001	36628 25008 70286
12	41 10166 33391	683 30420 30178	10882 33560 51137
13	6 10686 60380	120 49092 18331	2249 68618 68481
14	63025 24580	14 93040 04500	329 51652 81331
15	4523 29200	1 30874 62580	34 56159 43200
16	223 50954	8099 44464	2 60465 74004
17	7 41285	349 52799	14041 42047
18	15675	10 23435	533 74629
19	190	19285	13 89850
20	1	210	23485
21		1	231
22			1
G_n	5172 41582 35372	47486 98161 56751	4 50671 57384 47323

$r \backslash n$	23	24	25
1	1	1	1
2	41 94303	83 88607	167 77215
3	1 56863 35501	4 70623 00806	14 11979 91025
4	291 63425 74750	1168 10566 34501	4677 12897 38810
5	9641 68881 84100	48500 07834 95250	2 43668 49741 10751
6	99896 98579 83405	6 09023 60360 84530	37 02641 70000 02430
7	4 38264 19991 17305	31 67746 38518 04540	227 83248 29987 16310
8	9 74195 50199 00400	82 31828 21583 20505	690 22372 11183 68580
9	12 32006 88117 96900	120 62257 43260 72500	1167 92145 10929 73005
10	9 59340 12973 13460	108 25408 17849 31500	1203 16339 21753 87500
11	4 86425 13089 51100	63 10016 56957 75560	802 35590 44384 62660
12	1 67216 27734 83930	24 93020 45907 58260	362 26262 07848 74680
13	40128 25603 41390	6 88883 60579 22000	114 48507 33437 44260
14	6862 91758 07115	1 36209 10216 41000	25 95811 03608 96000
15	847 94044 29331	19582 02422 47080	4 29939 46553 47200
16	76 23611 27264	2067 71824 65555	52665 51616 95960
17	4 99169 88803	161 09499 36915	4806 33313 93110
18	23648 85369	9 24849 25445	327 56785 94925
19	797 81779	38807 39170	16 62189 69675
20	18 59550	1169 72779	62201 94750
21	28336	24 54606	1685 19505
22	253	33902	32 00450
23	1	276	40250
24		1	300
25			1
G_n	44 15200 58550 84346	445 95886 92948 05289	4638 59033 22299 99353

r \ n	26	27
1	1	1
2	335 54431	671 08863
3	42 36107 50290	127 08658 05301
4	18722 63569 46265	74932 90385 35350
5	12 23019 61602 92565	61 33820 71584 09090
6	224 59518 69741 25331	1359 80131 80050 44551
7	1631 85379 79910 16600	11647 57177 29112 41531
8	5749 62225 19456 64950	47628 83181 35563 36200
9	11201 51678 09551 25625	1 06563 27328 05417 95575
10	13199 55537 28468 48005	1 43197 07050 94236 05675
11	10029 07834 09984 76760	1 23519 41712 38300 92365
12	5149 50735 38569 58820	71823 16658 72819 82600
13	1850 56857 42535 50060	29206 89881 91531 09600
14	477 89861 83962 88260	8541 14923 18015 85700
15	90 44903 01911 04000	1834 63407 12628 48260
16	12 72587 72424 82560	294 06306 60708 24960
17	1 34373 17953 78830	35 56931 77639 22670
18	10702 55461 01760	3 27019 16252 10510
19	643 38390 18750	22926 84874 58010
20	29 06228 64675	1224 62963 12250
21	97591 04355	49 55640 56130
22	2389 29405	1 50155 51265
23	41 26200	3338 32005
24	47450	52 65000
25	325	55575
26	1	351
27		1
G_n	49631 24652 36187 56274	5 45717 04793 60599 89389

Have you read?

BOYER, LEE EMERSON, "=, Equal or Equals?" *The Arithmetic Teacher*, October 1955, pp. 91–92.

Did you ever get embroiled in an argument because of semantic differences? I'm sure you have and that you are also aware of the vocabulary problems confronting your students in mathematics. Mr. Boyer confines himself to one word—"equal"—plural or singular, take your choice. It appears simple on the surface, but check your colleagues and see if they are consistent. Your students can read this article and will also profit from it.

"Three plus two equals five."
"Three plus two equal five."

What do you think? Better take the test given in the article before you decide.

BURR, HARRIETTE, "Are We Providing for the Non-College Pupils in Mathematics?" *California Journal of Secondary Education*, November 1955.

How does it happen that 50 per cent of the high-school graduates can only reach sixth grade achievement in mathematics? If 55 per cent of the schools offer at least two semesters of non-college mathematics after grade nine, shouldn't the high-school graduate reach a higher level of attainment? The answer seems to be that the high-school students do not elect mathematics even if available. Over 50 per cent of the high-school graduates take no mathematics beyond grade nine.

What are the implications, and what should be done to change the existing situation? The author of the article gives some suggestions to publishers of textbooks, teachers, those who train teachers, school administrators, and counselors. This short article is thought-provoking, even without the statistical data to justify the cause.—PHILIP PEAK, *Indiana University, Bloomington, Indiana*

Edited by Lucien B. Kinney, School of Education, Stanford University, and Dan T. Dawson, School of Education, Stanford University, Stanford, California

Contributions of junior high schools to American mathematical education

by E. R. Breslich, University of Chicago, Chicago, Illinois

The history of American education shows that the schools have always responded to demands for adjustment of education to the needs of the community. At the time when the high schools came into existence the need was for leaders of society, i.e., doctors, ministers, and scholars. Accordingly, the major function of the high-school teacher was to train pupils for leadership. The curriculum was designed to prepare them for college. Entrance was made intentionally difficult by examinations designed to exclude those not qualified. This made the high-school population a rather select group of pupils. It is interesting to observe that mathematics was a major subject which everybody was required to take.

As the number of high schools increased, entrance was made easier. More and more pupils were admitted. They came from all classes of society, had widely different backgrounds, held different views as to the purposes of education, and varied greatly in ability. Failures were on the increase and the schools were being blamed. In response to criticisms, curriculum revisions were undertaken. The important aim now was to prepare pupils for citizenship. In the new curriculum mathematics became less important, and the popularity of the subject declined.

The teacher now gave most attention to the large group of pupils of average or below average ability. The result was that the gifted child who had comparatively few difficulties was allowed to shift for himself and was being neglected. He became bored. Gradually he lost interest.

Recently attention has been called by leading educators, scientists, and industrialists to the fact that there is again a need for examining the curriculum. To the teacher of mathematics the following criticisms are of special interest:

1. Sufficient attention is not being given to pupils of exceptional ability.
2. Steps should be taken to reduce or eliminate the present shortage of mathematically educated persons, since such persons are greatly needed in research, science, industry, and engineering.
3. A revised program of instruction in the high schools should enable college students to acquire at least one year sooner than is possible at the present time the mathematics needed to do original work.

Since these suggestions are intended for the secondary schools, the junior high school as a unit in American secondary education should be expected to make contributions to the solution of these problems.

Special classes for mathematically gifted pupils

One method of giving more attention to the needs of those who possess mathematical talent, without neglecting the larger group of average ability, is to form special classes for them. The idea is not new in American education. It is being used successfully in a number of school systems. Such classes are usually small enough so that the teacher can be familiar with the progress of each pupil and can give individual attention when and where it is needed.

Some educators object to classification of pupils on the basis of ability. They fear that labeling pupils superior, bright, or gifted has a discouraging effect on the group of regular pupils. On the other hand, leaders in science and industry have voiced a growing concern about the loss of exceptional pupils at a time of increasing national and international competition. They are emphasizing the importance of high scholarship in mathematics and of early identification of pupils with mathematical ability.

The success of the plan of grouping pupils depends greatly on the method used to identify them. Certainly high intelligence is important, but a pupil with a high I.Q. is not always industrious or interested. He should not automatically be placed in a superior group. But a very industrious pupil with a lower I.Q. often succeeds in doing exceptionally good work. Thus, intelligence and industry are both important. Other traits to be considered are mathematical ability, persistence in overcoming obstacles, clear thinking, genuine interest, power of oral and written expression, and any other pertinent factors which the teachers of preceding courses can supply.

Moreover, when it is found that mistakes have been made in classifying, the plan should make it possible to transfer pupils from one group to another. Accordingly, when a pupil in the regular group shows marked ability he should be allowed to join the advanced group. There will be no complaint about grouping pupils if it is known that membership in the superior group is a recognition for outstanding accomplishment in mathematics.

Early guidance of junior-high-school pupils

Today's schools are being criticized for failing to attract more pupils to the study of mathematics. Leaders in science and industry repeatedly have called attention to this problem. They feel that it deserves more consideration. The junior high school should be able to contribute to its solution.

Much can be accomplished by careful guidance. A great deal of harm is done by misinformed persons who tell pupils that mathematics is a useless study, that it will do practically nothing for them in their future occupations, that they will probably never need it, and that modern industry provides instruction in the mathematics necessary for jobs. Such advice is unfair. It may deprive misguided pupils of chances to make a fair living in occupations that they like, but that are open only to those who have mathematical training. When they find later that they should have included certain work in mathematics when in school, the disappointment is great. Many have to take courses in evening schools or in correspondence schools, which is not only expensive, but seldom as satisfactory as work done in the regular schools.

Pupils should be advised correctly as to the uses and values of mathematics in present-day life. This is a responsibility of their teachers, administrators, parents, and friends. Moreover, guidance is best if it begins early, as in the junior high schools, instead of being left to the teachers of more advanced courses. Pupils need advice in the selection of the right mathematical program, in the choice of courses essential to such subjects as physics, chemistry, and biology, and in the choosing of mathematics for future careers that interest them.

Girls should understand that mathematics is no longer a boy's subject. During the last decades, adult life for girls has undergone great change. Industry now employs millions of women in technical jobs formerly regarded as work for men. Girls must prepare themselves to take over such jobs when the need arises. Counselors should therefore be well informed about the uses of mathematics today if they are to give proper guidance.

The cooperation of all those who advise junior high school pupils will do much to ease the present shortage of mathematically trained persons. Mathematical needs in the various occupations differ widely. In some trades the worker can get along without mathematics, but in countless others the skilled worker uses it often. Indeed, to many it is as much of a tool as the wrench, drill, and lathe. There are thousands of positions related to physics, chemistry, and engineering that are open only to those who know all the mathematics taught in the secondary schools and more. Also, a tremendous amount of research is being carried on in foods, drugs, science, and industry, all of which rests on a mathematical foundation. In the country's several thousand laboratories, tens of thousands of chemists, geologists, and meteorologists are engaged in research. They are continually making new and valuable discoveries. Year after year our economy is growing. But the number of persons qualified to carry on research has not increased in proportion because of the shortage of persons with the necessary mathematical education. Leaders in American industry warn us that if this continues we shall find competition in world markets increasingly keen; we may lose our present technical superiority; and our national security, peace, and happiness (which the findings of research insure) may be endangered.

Accelerating Preparation for Research

For several decades leaders in the field of mathematics have devised ways of eliminating waste in the study of that subject. As early as 1923, the National Committee on Mathematical Requirements proposed a curriculum which was expected to save the pupil's time without loss of the quality of his work. Today, leaders in industry again are emphasizing the need for eliminating loss of time in the study of mathematics. College sophomores, they find, would be able to do important research work, but must wait another year to complete the required mathematics.

Of course, classes for talented pupils should progress more rapidly than regular classes, but this group is too small to satisfy present-day needs. The junior high school, comprising grades seven to nine and being a part of secondary education, has great possibilities for starting the acceleration of the pupil's progress in mathematics. The following examples illustrate this.

Achievement tests in algebra and geometry disclose that one of the major difficulties of pupils is a lack of understanding of the fundamental concepts, principles, and processes. When the pupil's answer to $a^2 \cdot a^3$ is a^6 it is clear that the meaning of exponents is not understood. Again, when he writes $a/n+b/n$ is equal to $a+b/2n$ he evidently does not know the meaning of the fraction. One explanation of this lack of understanding is the method used in teaching. Pupils entering the ninth and tenth grades differ greatly in previous mathematical experiences. At that age some have acquired a considerable amount of the mathematics of everyday life, but the mathematical experiences of others have been very limited. Hence, the teacher must present the fundamental concepts as if they were new to all members of the class. This creates a very unfavorable teaching situation. Some pupils see nothing new in the presentation, find it uninteresting, and are bored. For others the progress is too rapid for assimilation. When, later in the course, new under-

standings based on understandings previously attained are to be developed, difficulties arise. Often it is necessary to stop all work until the deficiencies are removed. Loss of time therefore results.

The junior high school can avoid this. In the seventh and eighth grades the differences due to previous experiences are not great and the groups are more homogeneous. Students can discover new principles and processes by experimenting and observing under the guidance of the teacher. An abundance of geometry is available to him in his surroundings—lines, angles, triangles, squares, rectangles, cubes, spheres, and many other geometric figures. By observation he discovers their properties and relationships. These he verifies by drawing and measurement. Gradually he acquires the desired understandings of the concepts and relationships of plane and solid geometry.

At the same time he learns through use a considerable body of algebra—parentheses, literal numbers, signed numbers, exponents, simple formulas, and equations. Ample time is to be allowed for assimilation. Yet, some of the concepts cannot be completely understood at this stage. Definitions, if given at all, are always the last step in the process.

Work of this type makes time saving possible in the senior high school. The junior high school occupies a unique and important place in the mathematical education of the pupil. It requires that teachers be familiar with the attainment of elementary school pupils. For arithmetic must be continued. They also must be familiar with senior-high-school mathematics because the foundation is to be laid for future work. Time will be lost unless the high school can go on where the junior high school stopped. Continuity will eliminate waste and increase the pupil's chances for success in the secondary school and in college mathematics.

Have you read?

Dodge, Homer and Dodge, Norton, "What's Wrong with the Schools in the United States?—An Interview," *U. S. News and World Report*, October 7, 1955, pp. 116–120.

At the risk of being accused of using poor judgment, or worse, I suggest you read this article. It is the kind of an article which cannot be read without some form of violent reaction. You have heard that Russia is rapidly outdistancing us in the race to train scientists, mathematicians, and engineers. The Dodge father and son combination give their reasons for the existing conditions in this published interview. The interview took place on their return from an extended trip to Russia. You will be interested in their suggestions as to how we can obtain more scientific people. You may be amazed at their evaluation of the role of the teacher. You will probably agree with some of the suggestions given for improving teaching. You will like the authors' attitude toward youth. Toward their philosophy of education, you will not be neutral. But read it, let your blood pressure rise, then as it recedes take a good look at the questions to see if you have the answers.

Gans, David, "An Introduction to Elliptic Geometry," *The American Mathematical Monthly*, Part II, LXII (August-September 1955), pp. 66–67.

Many people feel that the non-Euclidean geometries have never received their fair share of attention either in the training of teachers or in the teaching of high-school students. This is a brief, well-written article, which not only presents some of the historical developments, but also some of the concepts found in these geometries. The approach to both the double and single elliptic geometries is intuitive. Your students can read the article and will want to read more. They will be aroused to further thought by statements to the effect that: line and straight line are not synonymous; each pair of straight lines meet exactly twice; all straight lines bisect each other and have the same length. And they will certainly want to go deeper and find out more when they are told that: if two angles of a triangle are right angles the opposite sides are equal; a straight line is a closed line not intersecting itself.

The need for this article has existed for some time and I am sure you will put it on your *must* list.—Philip Peak, *Indiana University, Bloomington, Indiana*

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Department of Education, Brooklyn College, Brooklyn, New York

Men at work . . .

How do mathematicians work? Are their thought processes different from those of other thinkers? If so, in what way? These and kindred questions intrigue the layman and scholar alike. Unhappily, no very satisfactory answer is to be found in the rather scant literature. Such light as can be shed on the matter should be of considerable value to the teacher of mathematics.

A refreshing and suggestive discussion, regrettably all too brief, is to be found in a chapter by A. H. Read[1] entitled, "The Mathematician at Work," from which we quote a few excerpts in the hope that teachers will find them helpful as well as stimulating.

> One of the most common sources of difficulty to the student of mathematics is that a piece of work is very rarely presented to him in the form in which it was worked out by its originator. It has been condensed, polished up, and rearranged in a logical order, the ideas which gave it birth concealed like the works of a clock behind the clock-face. So does the mathematician appear as a super-mind because we cannot follow the trend of his thought.
>
> How does the mathematician think? It would be difficult indeed to give a coherent answer to this question, but something may be done to dispel popular illusions. One might begin by saying that he must be possessed by an overriding curiosity. In mathematics it is almost more important to be able to ask questions than to be able to answer them. "Who dragged whom, how many times, in what manner, round the wall of what?" quoted Jeremy Stickles in *Lorna Doone:* surely the propounder of this question was a bit of a mathematician.

In the next excerpt our author frankly comes close to begging the question, but perhaps no more so than do others striving toward the same goal:

> One of the distinguishing characteristics of the true mathematician is that, out of the multitude of questions that occur to him, he is able to select the ones that are worth answering. We have seen some of the preferences which guide him in his choice, the desire for generality, the pursuit of rigour and precision. The real criterion is whether the study of the problem is likely to be fertile in fresh ideas and to give birth to elegant mathematics: but this is a little like the advice to the Snark-hunters that they should recognise the creature by its taste. The great mathematician, like the great performer in most human activities, has a kind of intuition in such matters.
>
> It is nevertheless possible to single out one or two typical questions that he may ask. In modern mathematics, for instance, we have learnt that when a problem has defied repeated attempts at solution we ought to ask, can it be *proved* that the problem is insoluble?

Allusions are made to certain classical problems which were eventually shown to be insoluble—squaring the circle, duplicating the cube, trisecting an angle, the proof of parallel postulate, the algebraic solution of the general quintic. He then goes on:

> It is rarely the case that the proof of the impossibility of solving a problem finally ends the matter. Out of the ashes of the old problem a new one arises. We know from the Fundamental Theorem of Algebra that the quintic equation should have five solutions, and if the solutions cannot be expressed by means of the ordinary operations of algebra, how *can* they be expressed? If Euclid's parallel postulate cannot be proved, what happens in a geometry in which it is denied?
>
> This last question is typical of the modern axiomatic mathematics. If the axioms cannot be

[1] *A Signpost to Mathematics* ("Thrift Books," No. 8 [London: C. A. Watts & Co., Ltd., 1951]), chapter 8. Quoted by permission.

deduced from each other, what happens if we change one of them? For instance, what happens if, instead of assuming the axiom $ab = ba$, we substitute for it $ab = -ba$? This is a question which the pure mathematician may legitimately put to himself, but it so happens that the new sort of algebra which results, has important applications in applied mathematics. . . . We here see how the exacting and, as it might seem, tiresome process of delving down to the roots of the subject can bear fruit in the form of new and imaginative branches of mathematics.

Here he comes perhaps closest to the nub of the matter—the best advice on how to solve a problem is to "go on asking questions about it." What kind of questions? Well, one question to ask is: What previous problem or theorem involved the same kind of thing as this one? Or can the problem be restated in a different way? Or again: Do the given conditions admit of any simple deductions, which in turn may suggest a procedure or a solution? Try the supposition that the required result is false. Finally, are there any special cases of the problem? (This is especially helpful in connection with loci problems.) "It may sound strange that a mathematician bent upon new discovery is chiefly concerned with *proving* things rather than *finding* them. But this is on the whole the way in which mathematics progresses. The mathematician, having posed himself a question, *feels* his way to an answer and then sets out to prove it to be right." (Italics ours.) By way of summary, Mr. Read exhibits commendable humility concerning our insight into mental processes:

There is perhaps not such a gap between mathematical and other forms of thought as exists in popular imagination. Of course, the great in all walks of life have a way of their own, and it would be presumptuous to attempt to penetrate further and say what it is that distinguishes the truly great mathematical mind. Even our own mental processes are largely a mystery to us. We cannot say what happens to us in the moment of enlightenment, or the moment when against probability we notice the clue which turns out to be the essential link in the chain. All that we can say is that, if we have thought about a problem, particularly if we have asked ourselves sensible questions about it, the solution will often come to us easily when we return to it after a period of leisure, or even will flash upon us at a moment when we are occupied with other things. And this is a phenomenon which appears to be as common in the great discoveries of mathematics and science as in our attempts to remember our next-door-neighbour's sister's married name.

In conclusion, it seems to me that at least three desirable pedagogical changes are implied by the foregoing:

1. Paying less homage to the current popular fashion of teaching mathematics by manipulation, over-emphasizing experimental, inductive procedures, unhappily giving the impression that mathematics is largely a physical science.
2. Greater emphasis, informally, on mathematical induction; specifically, fuller use of symbolism, generalization, and the principle of permanence of form in teaching elementary algebra.
3. Greater flexibility in the use of the techniques of deductive logic; specifically, explicit attention given to such matters as the nature of postulates and definitions, converses, inverses, law of contrapositives, Euler's circles, and other aspects of elementary classic logic, drawing freely upon additional non-mathematical content.

Readers interested in ramifications of some of the matters raised here may find the following bibliography of some help:

BIBLIOGRAPHY

DADOURIAN, H. M. *How to Study, How to Solve: Arithmetic through Calculus.* Cambridge, Mass.: Addison-Wesley Press, 1951.

DAVAL, R., and GUILBAUD, G. T. *Le Raisonnement Mathématique.* Paris: Presses Universitaires de France, 1945.

HADAMARD, JACQUES. *An Essay on the Psychology of Invention in the Mathematical Field.* Princeton University Press, 1945, 1949.

HARDY, G. H. *A Mathematician's Apology.* Cambridge University Press, 1940.

POLYA, G. *How to Solve It: A New Aspect of Mathematical Method.* Princeton University Press, 1945.

WERTHEIMER, MAX. *Productive Thinking.* New York: Harper and Bros., 1945.

Further note on the printing of mathematics . . .

Not long ago,[2] this column made some observations on the problems involved in the art of printing mathematics. We now return to the matter, pointing out that the nature and effectiveness of mathematical notation, on the one hand, and the mechanics and intricacies of the printing process on the other, are indeed subtly related. As the astute Augustus de Morgan observed over a century ago:[3]

> Mathematical marks or signs differ from those of written language in being almost entirely of the purely abbreviative character. . . . Too much abbreviation may create confusion and doubt as to the meaning; too little may give the meaning with certainty, but not with more certainty than might have been more easily attained. . . .
>
> The subject of mathematical printing has never been methodically treated, and many details are left to the compositor which should be attended to by the mathematician. Until some mathematician shall turn printer, or some printer mathematician, it is hardly to be hoped that this subject will be properly treated.

Fortunately the subject has now been adequately presented by two recent works, one of which we have previously reported on.[4] The other is a slightly more modest, yet equally scholarly monograph, put out by a well-known American press.[5] Despite its brevity, this intensely practical brochure is designed to help authors, editors, and others concerned with the preparation and economical production of books and articles containing mathematical expressions. The "Preface" to this work makes the following point:

> Nothing—or almost nothing—is impossible in this field. Symbols in great variety and in extreme range of size and position relationship are available. However, when variety and range exceed rather strict mechanical limits, economic forces are set in motion which tend to limit the ultimate market for books and to reduce the number of pages published in journals.

The book is written from a printer's point of view. It describes how manuscripts are converted into type, and suggests how costs may be limited and controlled. The intent is neither to limit authors in their use of symbols and notational structure, nor to discourage typographic treatments necessary at a given pedagogic level, but rather, through analysis of styles and forms, to provide a basis for good judgment in order to assure successful publication.

It is pointed out that the chief factors affecting the difficulty and expense of mathematical composition have to do with three qualities of the original manuscript: the variety and nature of the symbols used in the notation; the justification required by the notation; and the condition of the copy. Each of these is discussed in turn, as are the various methods of composition, problems of setting and spacing, kinds and sizes of type, and variations in style both in manuscript and in type.

To us, a very brief section on "the eccentricities of letters" proved very illuminating:

> When spaced in file as words, and formed in mass as sentences the lead soldiers of our alphabet have little individuality. In reading them our eyes scan familiar groupings and project completed word or phrase images to the mind.
>
> In their mathematical usage letters are separated from their customary groupings and thrown into arbitrary and unfamiliar combinations. We discover, or rediscover, that here are 26 distinct individuals, severally tall, short, fat, and thin. . . . As a source of *symbols*, the alphabet is ideal for notation purposes; as a source of *forms*, it is unhappy.

The implications of these observations have many ramifications.

Perhaps many readers will find greatest interest in the section devoted to the preparation and marking of a manuscript. Many of the suggestions given have been taken from an earlier pamphlet, *Author's*

[2] The Mathematics Teacher, XLVIII (March 1955), pp. 165–166.

[3] *Penny Cyclopaedia (1842)*, vol. 23, article, "Symbols and Notation," pp. 442 ff.

[4] T. W. Chaundy, P. R. Barrett, and Charles Batey, *The Printing of Mathematics* (London: Oxford University Press, 1954).

[5] *Mathematics in Type* (Richmond, Virginia: The William Byrd Press, 1954), 58 pp.

Manual, printed by the *Duke Mathematical Journal*. Indeed, among the several collaborators responsible for the monograph under discussion we find that J. M. Thomas, formerly editor of the *Duke Mathematical Journal*, played a leading part, as did Wade H. Patton of the Lanston Monotype Machine Company.

In concluding our observations concerning this significant contribution to the field of printing mathematics, we should like to recall the sentiment of the Spottiswoode Committee on Mathematical Notation and Printing (British Association, 1875):

> Anything which tends towards uniformity in notation may be said to tend towards a common language in mathematics; and whatever contributes to cheapening the production of mathematical books must ultimately assist in disseminating a knowledge of the science of which they treat.

And as the collaborators of the present work themselves conclude:

> We bespeak a middle course between these two statements of fact: "All mathematical thought can theoretically be printed in a single roman font with no special characters whatever" . . . and . . . "'such trumpery of tricks of abbreviation' . . . have led to the creation of a language so powerful that it has actually itself become an instrument of research which can point the way to future progress."

On the mathematical training of navy personnel . . .

Many of our readers will doubtless recall the provocative letter of Admiral Nimitz, written in the early part of World War II, dealing with the mathematical training for officer candidates in the United States Navy.[6] By way of comparison, we reproduce portions of a letter by Isaac Newton. Having been asked to do so, he criticizes the mathematical curriculum in vogue at Christ's Hospital in 1694, especially as it applied to boys preparing to go to sea. After pointing out certain defects in the courses of study then offered, Newton suggests some serious shortcomings and omissions, as indicated by the following excerpts:

> But then for the things it conteins I account it but mean and of small extent. It seems to comprehend little more then the use of Instruments, and the bare practise of Seamen in their beaten road, w^ch^ a child may easily learn by imitation, as a Parrot does to speak, w^th^out understanding in many cases the reason of what he does; & w^ch^ an industrious blockhead, who can but remember what he has seen done, may attain to almost as soon as a child of parts, and he that knows it is not assisted thereby in inventing new things & practises, and correcting old ones, or in judging of what comes before him: Whereas the Mathematicall children, being the flower of the Hospitall, are capable of much better learning, & when well instructed and bound out to skilfull Masters, may in time furnish the Nation w^th^ a more skilfull sort of Sailors, Builders of Ships, Architects, Engineers and Mathematicall Artists of all sorts, both by Sea and Land, then ffrance can at present boast of. The defects of the old scheme you may understand by these instances.
>
> 1. It conteins nothing more of Geometry than what Euclid has in the beginning of his first book, and in the 10^th^ and 12^th^ propositions of his sixth booke. All which is next to nothing.
> 2. There is nothing at all of symbolical Arithmetick, w^ch^ tho' not requisite in the vulgar road of Seamen, yet to an inventive Artist may be of good use.
> 3. The taking of heights and distances, and measuring of planes and solids is alsoe wanting, tho of frequent use.
> 4. Nor is there any thing of spherical Trigonometry, tho the foundacon of a great many usefull Problems in Astronomy, Geography and Navigation.
> 5. Neither is there any thing of Sayling according to the severall Hypotheses, nor of Mercators Chart, nor of computing the way of Ships tho things w^ch^ a Sailor ought not to be ignorant of.
> 6. The finding the difference of Longitude, Amplitude, Azimuts and variation of the compass is alsoe omitted, tho these things are very usefull in long voyages, such as are those to the East Indies, and a Mariner who knows them not is an ignorant.
> 7. Nor is there one word of reasoning about force and motion, tho it be the very life and Soul of Mechanical skill and manual operations and there is nothing soe Mechanical as the frame & managem^t^ of a ship. By these defects it's plain that the old scheme wants not onely methodizing, but alsoe an enlargem^t^ of the learning. ffor some of the things here mentioned to be wanting, are requisite to make a Mariner skilfull in the ordinary road, and the rest may be often found usefull to such as shall become eminent for skill & ingenuity, either in Sea affaires, or such other mechanicall offices and imployments, as the

[6] *American Mathematical Monthly*, XLIX (1942), p. 212.

King may have occasion in his Yards, Docks, fforts, and other places, to intrust them with.

Now the imperfections of the whole scheme are pretty well supplyed in that new one w^ch is proposed to be established. ffor this is methodical, short & comprehensive. It excells the old one beyond comparison; I have returned it to you, w^th some few alterations for making the affinity, coherence and good order of the severall parts of the learning, more cleare and conspicuous, & supplying some defects. The alterations are of noe very great moment, excepting the addition of the last Article, w^ch conteins the science of Mechanicks. The rest is as perfect as I can make it without this Article. whether this should be added may be a question, but since you concur w^th me in the affirmative, I'le set downe my reasons for the addition. ffor w^thout the learning in this Article, a Man cannot be an able and Judicious Mechanick, & yet the contrivance & managem^t of Ships is almost wholly Mechanical. Tis true that by good naturall parts some men have a much better knack at Mechanical things then others, and on that acco^t are sometimes reputed good Mechanicks, but yet w^thout the learning of this Article, they are soe ffarr from being soe, as a Man of a good Geometrical head who never learnt the Principles of Geometry, is from being a good Geometer. ffor whilst Mechanicks consist in the Doctrine of force and motion, and Geometry in that of magnitude and figure: he that can't reason about force and motion, is as far from being a true Mechanick, as he that can't reason about magnitude and figure from being a Geometer. A Vulgar Mechanick can practice what he has been taught or seen done, but if he is in an error he knows not how to find it out and correct it, and if you put him out of his road, he is at a stand; Whereas he that is able to reason nimbly and judiciously about figure, force and motion, is never at rest till he gets over every rub. Experience is necessary, but yet there is the same difference between a mere practical Mechanick and a rational one, as between a mere practical Surveyer or Guager and a good Geometer, or between an Empirick in Physick and a learned and a rational Physitian. Let it be therefore onely considered how Mechanical the frame of a Ship is, and on what a multitude of forces and motions the whole business and managem^t of it depends, And then let it be further considered whether it be most for the advantage of Sea affaires that the ablest of our Marriners should be but mere Empiricks in Navigation, or that they should be alsoe able to reason well about those figures, forces, and motions they are hourly concerned in. And the same may be said in a great measure of divers others Mechanical employments, as buildings of ships, Architecture, ffortification, Engineering. ffor of what consequence Mechanical skill is in such Mechanical employments may be known both by the advantage it gave the old to Archimedes in defending his City against the Romans, by w^ch he made himself soe famous to all future ages, and by the advantage the ffrench at present have above all other Nations in the goodness of their Engineers. ffor it was by skill in this Article of learning that Archimedes defended his City. And tho the ffrench Engineers are short of that great Mechanick, yet by coming nearer to him then our Artificers doe, we see how well they fortify and defend their owne Cities, and how readily they force and conquer those of their Enemies. You may consider to what perfection that Nation by their Schooles for Sea-Officers had lately brought their Navall strength, even against all the disadvantages of nature, and yet your schoole is capable of out-doeing them, ffor their's are a mixture of all sorts of capacities, your's children of the best parts selected out of a great multitude. . . .

And now I have told you my opinion in these things, I will give you Mr. Oughtred's, a Man whose judgment (if any man's) may be safely relyed upon. ffor he in his book of the circles of proportion, in the end of what he writes about Navigation (page 184) has this exhortation to Seamen "And if, saith he, the Masters of Ships and Pilots will take the pains in the Journals of their Voyages diligently & faithfully to set down in severall columns, not onely the Rumb they goe on and the measure of the Ships way in degrees, & the observation of Latitude and variation of their compass; but alsoe their conjectures and reason of their correction they make of the aberrations they shall find, and the qualities & condition of their ship, and the diversities and seasons of the winds, and the secret motions or agitations of the Seas, when they begin, and how long they continue, how farr they extend & w^th what inequality; and what else they shall observe at Sea worthy consideration, & will be pleased freely to communicate the same with Artists, such as are indeed skilfull in the Mathematicks and lovers & enquirers of the truth: I doubt not but that there shall be in convenient time, brought to light many necessary precepts w^ch may tend to y^e perfecting of Navigation, and the help and safety of such whose Vocations doe inforce them to commit their lives and estates in the vast Ocean to the providence of God." Thus farr that very good and judicious man Mr. Oughtred. I will add, that if instead of sending the Observations of Seamen to able Mathematicians at Land, the Land would send able Mathematicians to Sea, it would signify much more to the improvem^t of Navigation and safety of Mens lives and estates on that element.

I hope S^r You will all interpret my freedome in this Letter candidly and pardon what you may therein think amiss, because I have written it with a good will to your ffoundation, and now I have spoke my thoughts I leave the whole business to the wisdome of your selfe and the Governors. I Am,

Hon^rd S^r.

Your most humble & most obedient Servant,

Cambridge May 25^th, 1694. Is. Newton.

A column of unofficial comment

Some thoughts on teacher education

*by John R. Mayor, Director, Science Teaching Improvement Program,
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Nearly all scientific organizations have recently established committees on teaching. Some have definite programs already in operation, and others are studying the problem of improving the quality of science teaching, with the hope that a conclusion may be reached as to the best kind of activities to undertake. The deep concern of scientists, engineers, and scientific organizations concerning science teaching, results from the present and anticipated shortage of scientific personnel. This shortage promises to be as critical in the area of science teaching—perhaps even more so—as in any other area. Interest in science teaching has greatly increased during the past five years due to widespread criticism of science programs in secondary schools and criticism of secondary-school programs in general. Many scientists realize that, during the past twenty-five years, they have neglected a responsibility which now must be met.

In physics, chemistry, and biology, as well as in mathematics, education committees of scientific organizations are concerned with modification of curricula. Scientists would like to see modern mathematics and science occupy a more important place in secondary-school courses. It is recognized that many teachers have not had much opportunity to keep current with modern scientific developments because of heavy responsibilities as teachers and because universities have failed to offer graduate courses in science and mathematics that are appropriate for teachers. There appears to be a growing movement toward making such courses available. The summer sessions of 1956 will see a considerable increase in number of courses in science and mathematics offered at the graduate level for graduate credit for teachers.

Good undergraduate preparation of teachers is closely related to this problem. In his undergraduate program, a teacher of science and mathematics needs to learn more about modern developments in his general area. The teacher of mathematics would profit from knowledge of the most recent developments and trends in chemistry, physics, and biology, as well as in mathematics. Necessarily, a very great demand is made on the time of the prospective teacher during his four years of undergraduate study.

During the past five to ten years there has been an increasing tendency in schools of education to provide and require a block-semester of student-teaching. The term "block-semester" is used to indicate that the fifteen or sixteen hours of work carried by the teacher during that semester will all be centered around the student-teaching experience. Under this plan, the student-teacher is free to go to another community to engage in full-time teaching for eight to ten weeks. The advantages of this kind of student-teaching experience

to the undergraduate prospective teacher are fairly obvious. In general, student-teachers in these programs appear happy with this arrangement and later feel that the experience in the off-campus community has been exceedingly valuable.

It must be borne in mind, however, that the undergraduate, if he is to be well-prepared in modern mathematics and science, already has too many demands made on his time. The value of the block-semester plan must be weighed against the desirability of using the time to learn more about modern developments in the field in which he may eventually teach. The traditional pattern of teacher education, which involves assignment to classes in a university or city high school, also has its advantages. Devoting a longer period to student-teaching, which interferes—especially in the preparation of mathematics and science teachers—with adequate subject-matter experience, is a course open to serious question in a four-year teacher-education program. The block plan actually shortens the eight semesters of undergraduate work by one semester.

The wide concern of scientists with the teacher shortage and teacher education has brought frequent criticism of requirements in professional education. Justification can be given for the usual requirement of fifteen to twenty semester-hours in professional education. If more time must be devoted to student-teaching, then it probably should be taken from courses in child development, schools and democratic society, or even educational psychology.

My own preference is that the requirements in professional education should not be increased, even by the provision of extending time for student-teaching in off-campus schools.

This is a difficult choice for those responsible for mathematics teacher-education programs. But it is a problem that must be faced squarely by all of us who wish to make the most of teacher-education opportunities so that science teaching in our secondary schools will be as strong as the needs of our time demand.

Education in a Colombian village

It is eight o'clock in the morning in the small Colombian village of Zarzal. The school day opens with the collective recital of the Roman Catholic catechism. As the pupils arrive they take their seats, open their copies of the catechism, and begin reading aloud. The teacher may call the class to order almost any time after eight o'clock. Catechisms are put away. The roll is taken and absences are noted. There is no rigid schedule to regulate the morning and afternoon sessions: the teachers may vary their presentations as they see fit. Usually, however, one grade is put to work copying sentences or pictures into notebooks, while the other is put through an oral question-and-answer exercise.

Counting sticks and an abacus are used to teach arithmetic. Simple addition, subtraction, multiplication, and division are demonstrated by the teachers on the blackboards. Pupils are given elementary problems to solve in their notebooks, and they copy and memorize multiplication tables. The teachers drill their pupils on the "tables" in the following manner:

Teacher: "What is six times six?"
Pupils (in chorus): "Thirty-six!"
Teacher: "What is eight times five?"
Pupils: "Forty"
Teacher: "Is nine times nine eighty-three?
Pupils (indignantly): "No!"
Teacher: "Is it seventy-one?"
Pupils (more indignantly): "No!"
Teacher: "Is it eighty-one?"
Pupils (triumphantly): "YES!!!"

Timid pupils or those uncertain of the answer either say nothing or are drowned out by the others. The teacher may drill these individually when he believes they need special help. Wrong answers are greeted with howls of derisive laughter. The class dunce is generally made to feel miserable.—*From a report,* Formal Education in a Rural Colombian Community, *Office of Education, U. S. Department of Health, Education and Welfare.*

Reviews and evaluations

Edited by Richard D. Crumley, University of South Carolina, Columbia, South Carolina, and Roderick C. McLennan, Arlington Heights High School, Arlington Heights, Illinois

BOOKS

Analytic Geometry. Neal H. McCoy and Richard E. Johnson, Rinehart and Co., New York, 1955. Cloth, xiv+301 pp., $4.00.

For the most part this is another of a long list of texts for analytic geometry following the traditional pattern. The topics treated are the usual ones, and their order of presentation conforms to recent practice. For example, translation of axes is treated before conics with vertex or center at an arbitrary point, and so this idea is used to derive the standard forms for these conics. The student is thus given an immediate practical application of this valuable tool. Determinants are not discussed, and the last chapter of forty-two pages is devoted to solid analytic geometry.

Proofs are sufficiently rigorous but should be readable for students. They are detailed but not cluttered. New topics are carefully introduced and are followed by adequate examples with figures which are better than average for a text of this type. In contrast to other analytic geometry texts on our shelves, both the figures and print are large enough to be read without eye strain.

Sketching the hyperbola in polar coordinates could be made to depend more on an analysis of the function and less on the memorization of certain formulas. In fact, it is the reviewer's opinion that sketching the conics can be made less dependent on formulas than is currently the practice, the present text being included. The authors make a good start in this direction with a chapter on equations and graphs, but they do not exploit to the fullest the ideas there introduced. Under the development used in this and other texts students become confused by the number of formulas necessary to work the somewhat artificial exercises—artificial in the sense that they offer only more drill on the formulas: $e=c/a$, $l.r.=2b^2/a$, $a^2+b^2=c^2$, and $a^2-c^2=b^2$. Perhaps the main difficulty is that the letters used are not the least bit descriptive in light of what they represent in the figures. The criticisms in this paragraph apply to geometry texts in general and not only to the one presently being reviewed.

More attention could have been paid to points of intersection of curves if this topic had been placed after the discussion of conic sections. The ideas of calculus are nowhere used, but formulas for the tangents to the conics are derived.

Students will find particularly helpful the section on points of intersection of curves in polar coordinates and the chapter on algebraic and transcendental curves. A more than adequate selection of exercises is available.

The text can be recommended to anyone wishing a text for the traditional course in analytic geometry.—*Arnold Wendt, Western Illinois State College, Macomb, Illinois.*

Arithmetic in General Education, Dewey C. Duncan, William C. Brown Co., Dubuque, Iowa, 1955. Paper, vi+194 pp., $2.25.

This is a comprehensive and condensed study of arithmetic. The author, who is a junior college teacher, wrote it for use in a semester course in pre-algebra arithmetic review.

In this reviewer's opinion this book could not be used in the ordinary high school general mathematics classroom. The language is somewhat beyond the scope of the average high school student. For example, on page 22, " . . . subtract arithmetically the absolute value of the number of smaller absolute value from the absolute value of the other. . . . " On page 25, " . . . algebraic addition and algebraic subtraction are more comprehensive than arithmetical addition and arithmetical subtraction; . . . " On pages 44–45 is found an *algebraic* proof of the fundamental property of fractions. Also the scarcity of practice exercises would make it impractical for use in the high school, where drill materials are necessary to fix the ideas.

The material contained in this work is of such nature that it would be a valuable addition to the mathematics teacher's library. The explanations of the various mathematical ideas are clear and concise, and the author has attempted to include as many ideas as possible in the limited space he has allowed himself. Of especial interest is the appendix entitled "Arithmetical Miscellanea." Included in this unit may be found, among other things, the Sieve of

Eratosthenes, perfect numbers, Pythagorean numbers, Euler's formula, and the game of Nim.

A book of this type would be valuable to upper elementary and junior high teachers also. In fact, it could be used in schools of education for a semester course in arithmetic for elementary teachers.—*Carl T. Uth, Arlington Heights High School, Arlington Heights, Illinois.*

Fundamental Concepts of Mathematics (2nd edition), R. H. Moorman, Minneapolis, Burgess Publishing Company, 1954–55. Paper, iii+92 pp., $2.75.

The author explains in his preface to this second edition that the material in this text is designed for use "as an orientation into mathematics for poorly prepared college students in industrial arts, pre-professional curricula, and in general education."

Chapter I discusses "Number," Chapter II "Measurement," Chapter III "The Function Concept," and Chapter IV "The Nature of Proof."

One must bear in mind that this material has been prepared for presentation to students who may have had no high school algebra or geometry. Accordingly, lack of rigor may be forgiven. Many significant topics have been singled out for discussion, and these topics are treated in simple language so that, for the most part, the ideas should be accessible even to the poorly prepared student,

The most general criticism that might be levied against the text is that it includes many unacceptable statements which are passed off as definitions. We find on the first page that "mathematics is the science of relationships," "arithmetic is the science of relationships expressed in terms of numbers," and "algebra is the science of relationships expressed in terms of letters." The first problem in the first list of exercises asks the students to repeat these three definitions. There is little value in requesting students to memorize such collections of words. It would be better to explain early in the text that a fundamental principle of mathematics (and other disciplines as well) is that *a definition is worthless unless something can be done with it.* Adherence to this principle would necessitate much rewriting.

Chapter I includes a brief discussion of our number system and points out how bases other than *ten* might be used. In the section on decimal fractions it is shown that repeating decimals are fractions, but the converse is not pointed out. When irrational numbers are illustrated by $\sqrt{2}$, it is observed that, written as a decimal, $\sqrt{2}$ "will not come out exactly." But the important observation that $\sqrt{2}$ is not a repeating decimal is not made.

We find "$\frac{6}{0}$ = impossible!" A *definition* asserts, "Real numbers are any numbers which *do not contain square roots of negative numbers.*" The italics are not the author's. In general, the treatment of the complex number concept is most unsatisfactory.

Chapter II includes a discussion of approximate measurement and presents area and volume formulas. The measurement concept is associated with *ratio.* But the author fails to point out that a ratio is a quotient of two real numbers, hence a real number. A serious omission is the failure to describe clearly the geometrical process by which a line segment may be "measured" by a unit segment, i.e. a method for assigning a real number (length) to the given segment. Of course, this requires that real numbers be defined as infinite decimals. This has not been done.

In this chapter a confusing reference is made to Euclidean geometry. After observing that the Greeks were conscious of the fact that the sides of a rectangle could be incommensurable, so that no unit square would "fit exactly" an integral number of times into the rectangle, it is remarked that "Euclid showed how to deal with these incommensurable cases rigorously." The terms *length* and *area* do not occur in Euclid. These are number concepts and the Greeks had no "real number system." Measurement plays no role in Euclid's geometrical theorems. It is true that the definition of *proportion*, as used by Euclid and generally credited to Eudoxus, does enable one to establish the key theorems relating to similarity. The theorem that a line parallel to the base and cutting the sides of a triangle divides them proportionally is elegantly established in a proof which combines the commensurable and incommensurable case, but there is no suggestion of *measurement.*

An angle is defined as "the *measure* of a rotation." In the next breath the author speaks of measuring angles. Are we measuring a measure?

In Chapter III the function concept is illustrated graphically. Several basic concepts of statistics are presented. This should be of considerable interest to students. Direct and inverse variation are incorrectly defined in this chapter. A table which is supposedly *a complete outline of mathematical functions* is presented!

The last chapter on "The Nature of Proof" is rather well written. Inductive and deductive reasoning are described and contrasted. It is mentioned that there is a rigorous method of proof by induction applicable to some mathematical theorems. Unfortunately, this technique is not illustrated as might easily be done, say by establishing the formula for the sum of the interior angles of a polygon.

Syllogisms are discussed and the applicability of Venn diagrams to syllogistic reasoning is illustrated. Several theorems of geometry are carefully proved.

Analysis and synthesis as techniques of proof in geometry are contrasted. Unfortunately, the author leaves the impression that the word "analytic" in "analytic geometry" has

the original Grecian meaning. This latter "analytic" is generally taken to imply merely the introduction of a coordinate system.

Indirect proof is discussed and difficulties in applying logical laws to everyday reasoning are pointed out. A puzzling use is made of Bertrand Russell's paradox of the "barber who shaves every man who does not shave himself." The author implies that this paradox implies the falsity of the law of the excluded middle. Russell would be horrified at such an interpretation.

All in all the text can hardly be recommended for college use. With careful rewriting it could be made usable.—*Charles Brumfiel, Ball State Teachers College, Muncie, Indiana.*

BOOKLET

Developing Mathematical Literacy in Nebraska's Youth, Educational Services, Teachers College, University of Nebraska, Lincoln, Nebraska. Mimeographed booklet; 8½"x11"; 46 pp.; $.50.

Description: An introduction of eight pages gives the background of this report and lists the "29 competencies" from the Post-war Commission's Report. The introduction is followed by units to teach these competencies. Each unit is organized around the following headings: objectives, content, learning activities and techniques, and sources of information. These are the units included: Why study mathematics?; percents—ratio; estimating and rounding numbers; tables—graphs—statistics; measurement—nature, units and systems; the Pythagorean theorem; angles and geometric concepts; drawings, ratio proportion, similar triangles, and trigonometry; constructions—ratio—vectors; algebraic symbolism, formulas; the number concept; axioms; purchasing power; and proceeding from hypothesis to conclusion. It is stated that these units are intended to teach the "minimum requirement" for "mathematics courses, traditional and general" in Nebraska secondary schools.

Evaluation: The eleven people listed as contributors should be very much pleased with their results, although they label it "first draft–for criticism and experimentation." The consistency of their treatment and the wise middle-way of including a great deal of material without trying to write a textbook on how to teach mathematics are commendable features. It is to be doubted that the 29 competencies are to be revered so much that the whole concept of a "minimum requirement" be considered settled by quoting them. Each group needs to think through that concept and come up with its own answer. Although a good list, it can be improved by others and may not fit all situations. It will be interesting to hear the results of putting this booklet into operation. The booklet should serve as an inspiration for curriculum study groups all over the country.—*Henry Syer, Boston University, Boston, Massachusetts.*

DEVICES

Combination Drawing Instrument (No. 7551), W. M. Welch Manufacturing Company, 1515 Sedgwick Street, Chicago 10, Illinois. Plastic and metal instrument, $3.50 each.

Combining the functions of a compass, protractor, dividers, and T-square, this precision instrument is made of transparent vinylite with scales imprinted on the bottom of the body. A thin coating of vinylite protects the scales. A metal pivot point and an adjustable slide with an opening for a pencil point enable one to use the device as a compass. The adjustable slide has a positive-locking screw to hold it in place. The largest radius possible is 6¼ inches. Any one of three linear scales may be used in setting this radius: inches graduated to 1/32 inch, inches graduated to 1/50 inch, or centimeters graduated to millimeters. There are two angle scales for use as a protractor: 90° graduated to 5°, and a so-called "vernier" scale of 9° graduated to 15′. The drawing edges of the instrument are beveled.

This instrument is precision made and highly versatile. The functions performed best are: drawing arcs of exact radii, constructing angles to the nearest 15′ of arc, and measuring line segments accurately.—*Richard D. Crumley.*

C-Thru Architects' Scalemaster (No. AR-46); C-Thru Engineers' Scalemaster (No. EN-56), C-Thru Ruler Company, 827 Windsor Street, Hartford, Connecticut. Plastic rulers, 3¾"×6¾", with 14 and 9 scales respectively; list price for either, $7.20 per dozen.

Both rules have transparent, laminated scales and four slotted openings. The architects' rule has divided graduations for scales of 1½, 3, ½, 1, ⅜, ¾, ¼, ⅛, 3/32, and 3/16 = 1 foot and has also full divided half scale, inches to sixteenths, inches to thirty-seconds, and millimeters. The engineers' rule has graduations of 10, 50, 20, 40, 30, and 60 decimal parts to the inch, as well as inches to sixteenths, a millimeter half scale, and a millimeter scale.

These rules are well constructed and easy to read. They can be used effectively by students in constructing graphs, in drawing to scale, or in understanding ratio and proportion.—*Richard D. Crumley.*

Pattern Dial (No. 7552), W. M. Welch Manufacturing Company, 1515 Sedgwick Street, Chicago 10, Illinois. Plastic and metal instrument, $3.50 each.

This device consists of an eight-inch plastic disk with one radial slot and many tiny holes for a pencil point. In the center there is a metal pivot point. The edge of the disk is graduated clockwise in degrees from 0° to 360°. Five regular polygons (equilateral triangle, square, pentagon or star, hexagon, and octagon) are pictured on the dial. When pencil dots are made where the pictures are shown, accurate drawings of these polygons with diameters of 2" to 7" can

be made. Circles or arcs of radii from ½" to 3½" with ½" increments can also be drawn.

This device is well constructed and durable. Many students like to draw polygons and designs, so they will like this device. In doing this sort of work, they can discover some of the properties of polygons by themselves. The protractor function when added to the pattern-making function makes this device very useful. —*Richard D. Crumley.*

EQUIPMENT

Acu-Math Student Mannheim Slide Rule No. 400, Yoder Instruments, 142 North Market Street, East Palestine, Ohio. 10" plastic slide rule with leatherette sheath and instruction book; $2.00 each, quantity discounts.

Made entirely of fully cured white plastic, this slide rule has scales S, K, A, B, C1, C, D, L, and T engraved on its face. The cursor is made of clear lucite with a steel spring to prevent wobbling. A cardboard strip with 32 equivalents for various units of measure and appropriate settings accompanies the rule. This strip can be glued to the back of the slide rule or carried in the sheath. An eight-page instruction book also accompanies the rule.

This slide rule is very well constructed and operates efficiently. The similarity between this rule and professional models makes it an excellent rule for students.—*Richard D. Crumley.*

Brunton Pocket Transit (#5368), Keuffel & Esser Co., Adams and Third Street, Hoboken, New Jersey. Present price, $46.50 less educational discount.

This transit is actually pocket size being about 1"x3"x3". Even though it is small, it is versatile and easy to read. It consists of the following parts: compass, two levels, two protractors, vernier, mirror, and sight holes. The parts are mounted in a metal case that closes compactly. The transit combines the principles of a surveyors' compass, a prismatic compass, a clinometer, and a hand level. It can be held in the hand for many measurements or mounted on a Jacob's staff or on a tripod when especially accurate results are desired.

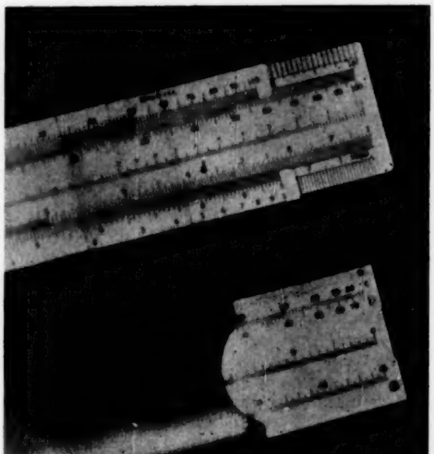
The instrument can be used to determine compass bearings, to measure horizontal or vertical angles, to run levels, and to measure the inclination of objects and per cent of grade or slope. Horizontal angles are measured by compass readings. A declination adjustment is provided for setting the compass circle for true or magnetic bearings. The cover has a three-place sine table. A booklet with complete instructions on its use accompanies the device. A leather belt carrying case ($4.50) and a ball-and-socket joint for mounting to a staff or tripod ($9.25) are also available.

Although this instrument may seem expensive, it is precision made of high quality material. Its ingenious design, which gives it great versatility, makes it useful for a wide range of field work. It is useful for junior high school classes as well as for more advanced classes.—*Donovan A. Johnson, University of Minnesota, Minneapolis, Minnesota.*

C-Thru 10" Slide Rule (Cat. No. 88), C-Thru Ruler Company, 827 Windsor Street, Hartford, Connecticut. Plastic slide rule with case and instructions; list price, $1.80 each.

Made of blue-white plastic, this slide rule has the A, B, C, C1, D, and K scales on the face side. The log, sine, and tangent scales are on the back of the ruler. The sliding rule has on its back decimal equivalents for the common fractions from 1/64 to 63/64, while the part of the rule covered by the sliding rule has twelve equivalents (with appropriate settings) for various types of units of measure. The sliding rule has at its two edges an inch scale and a metric scale. A vinyl case and a sheet of instructions accompany the rule.

One should not expect a precision-made slide rule at this price, but students may be disappointed in the accuracy of this rule. The cursor with the hairline on the model examined wobbled considerably so that only two-place accuracy was possible. Except for the cursor, the rule is cleverly designed.—*Richard D. Crumley.*



C-Thru 10" Slide Rule

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia

Organizing a mathematics club

by Annie John Williams, Durham High School, Durham, North Carolina

Soon after one begins teaching in a new school, he is likely to be asked to help with some extracurricular activity. Often, one is permitted to choose the activity he will sponsor. Not many schools have a mathematics club, yet it represents a real opportunity for a teacher to develop in capable students a keen interest in mathematics. Since the idea of a mathematics club is new to most people, one must be prepared to persuade administrators and students that membership in this club will be valuable. A mathematics club can be successful in either a junior or senior high school. If club membership can be limited, there will be an opportunity for many interesting projects, which would be very difficult to handle in a larger class group. Moreover, a club can help publicize the values of studying mathematics. In beginning a new club it is very important that the first year be successful, and every effort should be made to make that year the best possible.

Good parliamentary procedure can be learned in a mathematics club and should be stressed from the very beginning. Many schools have a student activities council of which all the clubs of the school are a part. If this is the situation, definite instructions for the organization of a new club will be given by the council and should be carefully followed. If the club is not a part of such a council, one should follow an acceptable form of organization, such as that outlined in Robert's *Rules of Order*. As soon as possible, a club should elect officers and write a constitution. This will be the first experience most of the members will have in writing a constitution. Therefore, they will need careful guidance. Among other points, the constitution should limit the number of members and set a standard for membership.

As soon as possible a project should be undertaken to show the club members, as well as other students in the school, what is being done in the new mathematics club. An excellent project and one that is very good to start with is the decorating of a mathematical Christmas tree. Have a large one and put it in the hall or cafeteria where it can be seen by all students. Geometric solids, polygons, and stars, when given typical Christmas covering, are lovely on a tree.

At first the students may be a little skeptical of making decorations. When they see some of the finished ornaments, they will no doubt decide that the ones they have made are prettier than the purchased variety. In junior high school, cones, cubes, pyramids, and prisms are suggested because they are easy to make. Of course spheres are good, but a little difficult for young students. In senior high, the solid geometry students can contribute sets of models that when painted or covered are lovely decorations. The five pointed star is good for either junior or senior high school and can be folded to give a three-dimensional appearance. One

can even use polygons that are easy to make. Cellophane straws, cut in small pieces and put together with elastic thread, will provide colorful and attractive designs. For the first year, a club may not be able to have star polyhedrons, but this is something to work toward. In making decorations, be sure to stress accuracy; the superiority of student-made decorations to ordinary decorations will then be appreciated.

There are places of interest in every locality for a club to visit. An industry that uses mathematics, a university or college mathematics department, a school of engineering, a bank or large office which uses calculating machines—all offer opportunities for visits. Such visits are usually practical for a small club, but not for large classes.

As in other clubs, there should be at least one social event during the year. This will provide an opportunity for members to get better acquainted with each other and with the adviser.

Have one or two outside speakers during the year. A college professor of mathematics who is interested in high-school students can be invited. The club might also select someone from industry, business, or engineering. If the members know something in advance about the speaker and are informed on his topic, much more will be gained. Not only will the entire club profit from hearing a speaker, but the officers will have a very valuable experience in conducting the meeting with an outsider present.

There will be other meetings that, although not as spectacular, will be just as important. The students will need a good deal of help in planning these meetings, and they should be encouraged to use as much of their own initiative as possible. These meetings will be good opportunities to develop individuals. Have students take part in the program. Try to have each club member on the program at least once during the year.

Members of the mathematics club that I sponsor have participated in the following programs with interest and satisfaction: Highlights in the History of Mathematics; Mathematical Puzzles and Games; The World Calendar; Modern Calculating Machines.

In many current advertisements there are lovely geometric designs, which can be made with curve stitching. Students can study the commercial designs and then make them for themselves. They enjoy learning how designs are made by stitching with thread, and they usually go on to use their originality in making designs of their own. In this, as with Christmas tree decorations, accuracy should be stressed.

Club interest may be stimulated by keeping a scrapbook. Pictures of members, programs, letters received by the club, samples of work, and the constitution are a few of the articles suitable for a scrapbook.

One of the greatest aids to a mathematics club is *The Mathematics Student Journal* published quarterly by The National Council of Teachers of Mathematics with the cooperation of the Mathematical Association of America. This journal is for high-school students. We should encourage students to read this publication and contribute material for publication.

The ideas I have suggested will enable you to make only a beginning. You will enjoy having a mathematics club, and you and your students will develop many ideas of your own to make your work valuable and interesting.

Edited by John A. Brown, University of Wisconsin, Madison, Wisconsin and Houston T. Karnes, Louisiana State University, Baton Rouge, Louisiana

The influence of the study of plane geometry on critical thinking[1]

by Carmen C. Massimiano, Pittsfield High School, Pittsfield, Massachusetts

In studies that have surveyed the purposes of teaching geometry, such reasons as, "to teach the students to think logically," and, "to understand the meaning of deductive proof," were most frequently mentioned. In one important survey that tabulated the opinions of 500 classroom teachers concerning the important objectives of teaching geometry, the one receiving the highest rating was, "to develop the habit of clear thinking and precise expression."

With allowances for varying phraseology and varying definitions or lack of them, all the literature indicated that the important values to be derived from the study of demonstrative geometry are related to an acquaintance with the nature of proof and to a familiarity with critical thinking as a method of thought.

The ability to think critically, as concerned in this study, involves three elements: (1) An attitude of being disposed to consider in a thoughtful way the problems and subjects that come within the range of one's experiences; (2) Knowledge of the methods of logical inquiry and reasoning; and (3) Some skill in applying these methods.

The chief problems of this study are: (1) Do high-school sophomores who have studied plane demonstrative geometry improve more in critical thinking than those pupils who have not taken geometry? (2) What is the relationship between achievement in geometry and critical thinking ability? (3) What is the relationship between achievement in all other major subjects during the sophomore year and critical thinking ability?

Special attention is given to the course of study in geometry as it is taught at Pittsfield High School. This course is organized around the objectives based on the exhaustive study of the Joint Commission of the Mathematical Association of America and The National Council of Teachers of Mathematics.

Relating geometric thinking to nongeometric situations is frequently employed as a means of motivating the logical emphasis of the course in demonstrative geometry. Since pupils learn subjects best in connection with their use in meaningful situations, the study of nongeometric exercises of logical reasoning that have relevance to geometric thinking facilitates the learning of geometric content, as in the treatment of the meaning of the definition, indirect proofs, fallacious reasoning, etc. At Pittsfield High School, a critical and appreciative attitude toward reasoning in everyday life is made to contribute to greater insight into geometric thinking.

[1] Abstract of Ph.D. Dissertation, University of Ottawa (Ontario, Canada).

On October 5, 1953, all 531 sophomore pupils at Pittsfield High School were given the *Watson-Glaser Critical Thinking Appraisal, Form AM*. During the following week, the sophomores were given *Form C* of the *Terman-McNemar Test of Mental Ability*.

In order to compare the "geometry" groups and the "non-geometry" groups, students from the two groups were paired on the basis of: (1) Intelligence as indicated by the verbal mental ability score on the *Terman-McNemar Test*; (2) Score on the *Watson-Glaser Test* administered on October 5, 1953; (3) Sex; (4) Age. On this basis three matched sets of paired pupils were constituted. On May 6, 1954, all sophomores were given *Form BM* of the *Watson-Glaser Critical Thinking Appraisal*.

The test scores on the *Terman-McNemar Test of Mental Ability* and the two forms of the *Watson-Glaser Critical Thinking Appraisal* as well as pupil-personnel records comprise the basic data of this investigation.

The answer to the major question, whether high school sophomores who have taken plane geometry for seven months improve more in critical thinking than those pupils who have not taken geometry, was first sought by noting the increase in the mean of the scores made in the two administrations of the *Watson-Glaser Critical Thinking Appraisal* and ascertaining whether such differences were significant.

The t distribution was used to test the significance of the differences between the two means. All groups tested made gains varying from 2.60 to 7.95. The gain made by the boys not taking geometry is significant at the low 20 per cent level; however, gains made by the other groups are highly significant at 0.1 per cent. Included in the groups that did make significant gains was one made up of girls in the commercial curriculum who did not take geometry.

The paired groups revealed a difference of 5.20 points between the mean gains of the boys who took geometry and those who did not take geometry, a difference of -0.31 points between the mean gains of the boys who took geometry and the paired group who took geometry and a foreign language, and a difference of 2.46 points between the mean gains of the girls who took geometry and those who did not. None of these differences is significant at the 5 per cent level. However, 5.20 reaches the 7 per cent level and 2.46 the 8 per cent level, so that both non-geometry groups did tend to increase less in the *Watson-Glaser Critical Thinking Appraisal* than the corresponding geometry groups—with probabilities of better than 1 in 10 that the differences were not due to chance. Consistent with the hypothesis, the difference of -0.31 between the two matched geometry groups (one taking geometry but no foreign language, the other taking geometry and French or Latin) had a probability of 90 per cent that the differences were only due to chance.

Only in group C-2 are the correlations sufficiently large to indicate a positive relationship between geometry grades and second *Watson-Glaser* test scores. In this group the second *Watson-Glaser* test scores and the difference in the *Watson-Glaser* test scores have about the same correlations with grades in other major subjects as with geometry grades (about .4 of .5). The ability to read adequately or to understand language discriminatingly may be a factor in the correlation with English grades. The inclusion of social-study-type material in the *Watson-Glaser Critical Thinking Appraisal* may be a factor with the correlation with United States history grades.

The data does not seem to support the theory that success in geometry (as measured by grades) is closely associated with critical thinking (as measured by the *Watson-Glaser* test scores).

Multiple correlation coefficients between grades in all major subjects during the sophomore year and the initial, final,

and difference in *Watson-Glaser Critical Thinking Appraisal* scores indicate that subject grades correlate more closely with the second *Watson-Glaser* test scores than with the first *Watson-Glaser* test scores or the change in *Watson-Glaser* test scores, and that subject grades and *Watson-Glaser* scores correlate most closely in the group of boys who took geometry and a foreign language.

Significant data for the regression coefficients, obtained by processing on an electronic computer, were consistent with the findings dealing with the correlations between geometry grades and the initial, final, and difference in *Watson-Glaser Critical Thinking Appraisal* test scores.

It is possible that the influence of studying geometry on the development of critical thinking ability (revealed in this study at an inadequate significance level) might be established at a higher level if the research in this study were extended to include additional numbers of students. This would seem a fruitful field of research, since quantitative data is needed to interpret current hypotheses as to the power of geometry, as customarily taught, to develop the ability to think critically in nongeometric contexts.

Let's look at language

Beginners in mathematics not infrequently wonder why the word "tangent" refers to two concepts which at first consideration present little, if any, relationship. In one connotation "tangent" refers to a line unrestricted in length. In another usage, "tangent" refers to a ratio expressed accurately to two, three, four, or more, digits. What can teachers of secondary-school mathematics do to help their pupils resolve this perplexity?

One approach, of course, is simply to invoke the basic principle that a word means just what we want it to mean—exactly this and neither more nor less. Thus a tangent to a circle may be defined in one of several ways, and the term "tangent" is arbitrary:

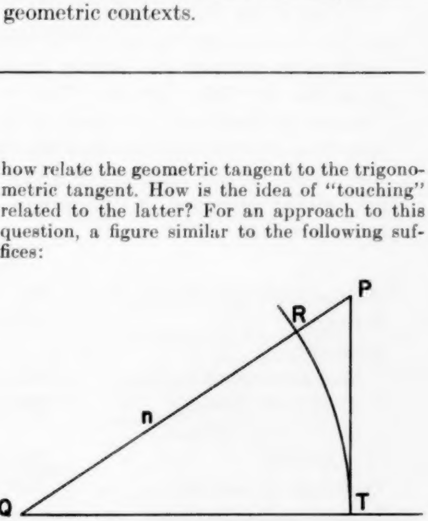
1. A line perpendicular to the radius of a circle at P, a point on the circle.
2. A line touching a circle at one and only one point.
3. A line which, though extended indefinitely, has but one point in common with the circle.

In like manner "tangent," when it refers to a trigonometric ratio, may be summarily defined in a way such as the following:

1. The ratio of the ordinate and the abscissa of a point (not the origin) on the terminal side of an angle placed in standard position.
2. The ratio of the side opposite an acute angle in a right triangle to the side adjacent to the acute angle.

Later the idea of a tangent to a circle may be extended to include tangency to any curve. The inherent idea of "touching," stemming from the Latin forebear *tangere*, shines clearly enough through the word "tangent" when it refers to geometric tangency.

Yet, despite whatever effectiveness the foregoing definitions may have, most pupils would probably feel more secure if they could somehow relate the geometric tangent to the trigonometric tangent. How is the idea of "touching" related to the latter? For an approach to this question, a figure similar to the following suffices:



Given: Right triangle QPT with $T=90°$.

By definition, the trigonometric tangent is:

$$\tan Q=\frac{PT}{TQ}$$

Draw: Arc TR with radius QT and center Q.

By definition PT, being perpendicular to TQ at T, is tangent to the circle with center Q and radius QT. This, of course, is the geometric tangent.

Also:

$$\frac{PT}{TQ}=\tan Q.$$

Hence, if the radius $QT=1$, $PT=\tan Q$. Accordingly, the trigonometric ratio PT/TQ equals the length of the geometric tangent PT, the segment between the arms of angle PQT.

From the foregoing considerations, the word "tangent" seems indeed quite appropriate.—*I. H. Brune, Iowa State Teachers College*

• NOTES FROM THE WASHINGTON OFFICE

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

Millions of pages for mathematics education

Recently the Publications Division of the NEA, in preparing its annual report, asked us to calculate the number of pages published by The National Council of Teachers of Mathematics during the fiscal year, June 1, 1954 through May 31, 1955. The results of our calculation, given below, rather surprised us. They may interest you too.

	Pages
New booklets and pamphlets	882,000
Reprints	133,000
Journals:	
MATHEMATICS TEACHER	7,870,000
Arithmetic Teacher	542,000
Mathematics Student Journal	700,000
Programs for meetings	770,000
Promotional materials (membership, sales, announcements)	353,000
Total	11,250,000

As you will note from the above figures, our major publication interest lies in our three journals, which account for more than nine million pages. THE MATHEMATICS TEACHER, as expected, used the largest number of pages, nearly eight million. Although the *Mathematics Student Journal* is small (4 pages), its large circulation produces an impressive total for number of pages printed. The importance of the *Arithmetic Teacher* to mathematics education is undoubtedly much greater than the comparative number of pages would indicate. With its frequency increased from four to six issues per year and with the subscription list rapidly growing, the presses should turn out at least one million pages of this journal during the new fiscal year.

Our second largest publication project is our yearbooks. The press run for a typical yearbook produces about two million pages. No new yearbook was produced and none was reprinted during the past fiscal year. The place of the yearbooks was partially filled by five new booklets and pamphlets. The number of pages listed is based upon the initial press runs only. It has already been necessary to go to press again on two of the booklets.

The programs for meetings were the printed programs for the Seattle, St. Louis, and Cincinnati meetings. The promotional materials included membership folders, lists of publications, and form letters and announcements sent out by the Washington Office.

All teachers of mathematics like problems. Here are some that you and your students might wish to work. We have simplified the data somewhat to make the computations easier. Remember that each two printed pages require one sheet of paper. Although the paper sizes and weights vary, the average size of a sheet was, if our own arithmetic is correct, about 7 by 10 inches.

About 9,400,000 pages were printed on 50-lb. book paper. The remaining pages were printed on 60-lb. book paper or other paper of equivalent weight. By 50-lb. weight we mean that 500 sheets measuring 25 by 38 inches weigh 50 pounds. Likewise 60-lb. weight means that the same amount of paper weighs 60 pounds. Thus the 60-lb. paper is of greater weight and thickness.

In a typical brand of English-finish paper, one-inch thickness of 50-lb. paper contains 332 sheets, while one-inch thickness of 60-lb. paper contains 277 sheets.

Using the above data, it should not be too difficult to solve the first three problems below. The fourth problem we leave for your thoughtful conjecture.

1. If all the sheets of paper could be laid end to end, how many miles long would the ribbon of paper be?
2. If the sheets of paper were stacked one upon the other, how high would the stack be?
3. What is the total weight of the paper?
4. If the pages could be equally divided among all the teachers of mathematics in the United States and all the material were read and applied, how much difference would it make in the teaching of mathematics?

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, The National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING

April 11–14, 1956
Schroeder Hotel, Milwaukee, Wisconsin
Margaret Joseph, 1504 N. Prospect Avenue, Milwaukee 2, Wisconsin

JOINT MEETING WITH NEA

July 2, 1956
Portland, Oregon
Lesta Hoel, Public Schools, Portland, Oregon

SUMMER MEETING

August 19–22, 1956
University of California, Los Angeles, California
Clifford Bell, University of California, Los Angeles 24, California

CHRISTMAS MEETING

December 27–29, 1956
Arkansas State College, Jonesboro, Arkansas
Lyle J. Dixon, Arkansas State College, State College, Arkansas

Other professional dates

Chicago Elementary Teachers' Mathematics Club

March 12, 1956
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Anne T. Linehan, O'Toole School, Chicago, Illinois

ANNUAL MEETING OF THE GEORGIA COUNCIL OF MATHEMATICS TEACHERS

March 16, 1956
Atlanta, Georgia
Marion Crider, West Georgia College, Carrollton, Georgia

A date to remember

April 11–14
The Thirty-fourth Annual Meeting
Milwaukee, Wisconsin

As required by the by-laws, notice is hereby given to the members of the National Council of Teachers of Mathematics that the Annual Business Meeting will be held at 4:30 P.M., Friday, April 13, 1956, at the Hotel Schroeder, Milwaukee, Wisconsin.

M. H. AHRENDT, *Executive Secretary*

Plan to attend the annual meeting of the Council in Milwaukee!

Dr. W. L. Duren, President of the Mathematical Association of America, will be one of the speakers at the 1956 annual meeting of The National Council of Teachers of Mathematics, which will be held from April 11 through April 14, 1956, in Milwaukee, Wisconsin. Dr. Duren has, for some time, been chairman of the important committee of the Association concerned with the undergraduate mathematical program.

Dr. W. A. Brownell, an outstanding authority in the teaching of arithmetic, and now Dean of the School of Education of the University of California at Berkeley, will address the group.

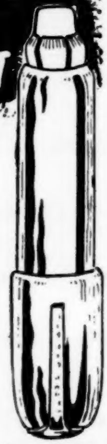
Forty other leaders in the field of mathematics and mathematics education have already accepted places on this program. Watch for complete details of the program in the March issue of THE MATHEMATICS TEACHER.

Wisconsin teachers of mathematics have planned a most cordial welcome including visits to local schools, tours of Milwaukee, and delightful social events.

Exhibits for the Thirty-fourth Annual Meeting

Teachers who wish to display classroom projects with a mathematical theme at the School Exhibits for the Thirty-fourth Annual Meeting of the NCTM, to be held in Milwaukee, April 11–14, may contact Miss Laura M. Wagner, 634 Short Street, Fort Atkinson, Wisconsin, for further information. An immediate response would be appreciated.

HANDS - Clean CLOTHES - Neat in spite of CHALK DUST



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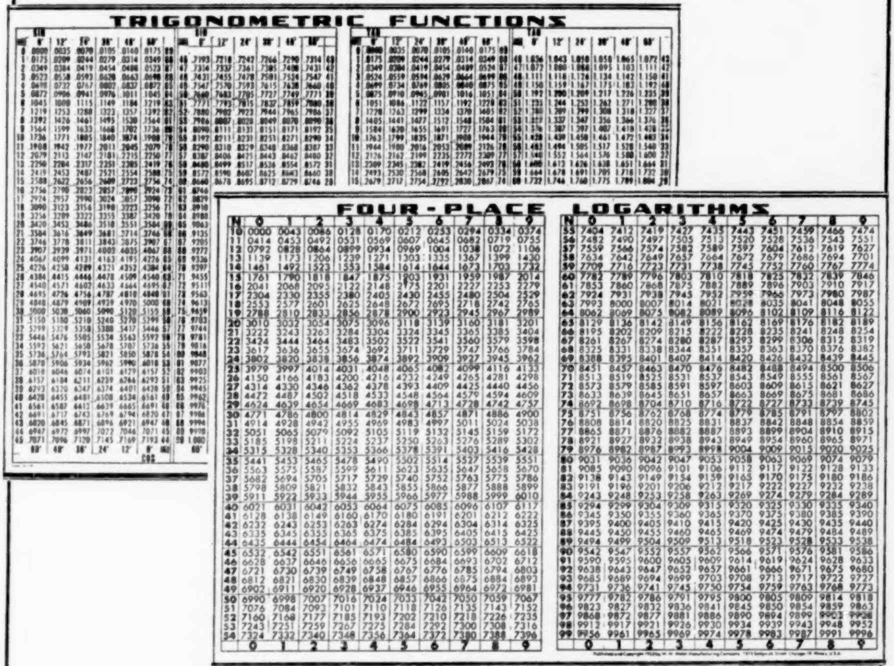

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